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DYNAMIC FORMULATION FOR GEOMETRICALLY-EXACT SANDWICH SHELLS

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Abstract—Extending our recent works on geometrically-exact sandwich beams and one-dimensional plates [see e.g. Vu-Quoc and Ebcioğlu (1995); Vu-Quoc and Deng (1995c); Vu-Quoc and Deng (1995a)], we present here a geometrically-exact sandwich shell theory, entirely in terms of stress resultants which accommodates finite deformations in membrane, bending, and transverse shear. The motion of the shell are referred directly to the inertial frame ; the transverse fiber of the sandwich shell has a motion identical to that of a chain of three rigid links connected by revolute joints. An important approximated theory is developed from the general nonlinear equations, the classical linear theory is recovered by the consistent linearization. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

The study of sandwich shells has been an area of increasing research interest, due to the enormous usage of such structures in modern mechanical systems in aerospace, automotive, ship-building and other industries. Numerous papers have been published on this subject. A review of papers on sandwich shells published before 1965 can be found in Plantema (1966). Later research on sandwich structures has been concentrating on accounting for more physical quantities in the model such as the transverse shear flexibility, the transverse normal strain, etc. The readers are referred to Noor and Burton (1989) and Noor, Burton and Bert (1996) and the references therein for the major contributions in this area. On the other hand, the dynamic modelling of the finitely deformable sandwich shells, especially when finite shear deformation is accounted for, is relatively unexplored.

The study on largely deformable single-layer shells, in a sense, can be divided into two major methodologies. One methodology, which has dominated the field for more than two decades, is the degenerated solid approach, in which the reduction to the resultant form is carried out numerically. We refer the readers to, e.g. Hughes (1987), Bathe (1996) for the comprehensive reviews, and also for Vu-Quoc and Mora (1989) and Vu-Quoc (1990) for related issues. The other methodology is based on geometrically-exact formulations, pioneered first for beams by Simo and Vu-Quoc (1986a), Simo and Vu-Quoc (1986b), Simo and Vu-Quoc (1986c), Simo and Vu-Quoc (1988), Vu-Quoc and Li (1995) and then for shells by Simo and Fox (1989), Simo, Fox and Rifai (1989), Simo, Fox and Rifai (1990) and Simo, Rifai and Fox (1990). In the geometrically-exact formulation, the reduction to the resultant form is carried out analytically, and thus leads to efficient numerical implementations and makes large-scale calculations possible.

Following the works on geometrically-exact sandwich beams and one-dimensional plates in Vu-Quoc and Ebcioğlu (1995) and Vu-Quoc and Deng (1995c), and the methodology in Simo and Fox (1989) in the formulation of the equations of motion for single-layer shells, we develop here the equations of motion for geometrically-exact sandwich shells, accounting for bending deformation, in-plane deformation, and shear deformation in each of the three layers. In the present theory, each layer can have arbitrary thickness, mass distribution, and different material properties. We adopt the single-director theory for each of the three layers, i.e. points that are originally located on a director must remain on the same director after deformation. The director, which is not extensible in the present formulation, must remain straight, but not necessarily perpendicular to the neutral surface of the corresponding layer of the sandwich shell ; thus shear deformations can be accounted

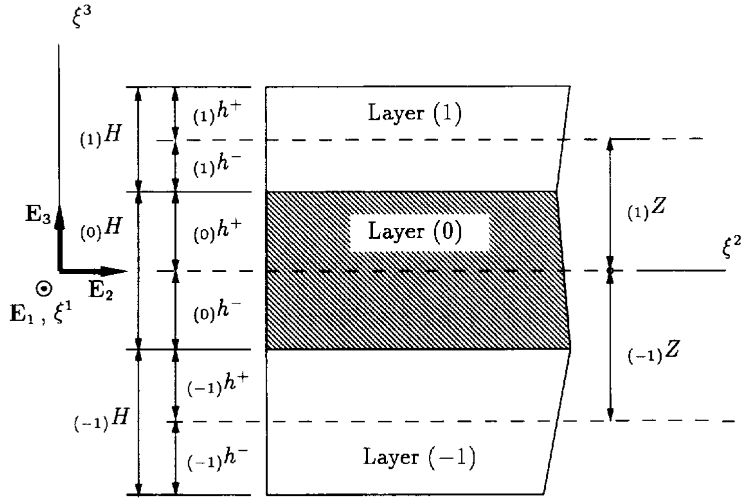

Fig. 1. Sandwich shell: profile and geometric quantities.

for. The motion of a typical transverse fiber can best be viewed as a chain of three rigid links connected to each other by revolute joints. The continuity in displacements across the layer boundaries is exactly enforced.

With the dynamics of the motion referred directly to the inertial frame, the equations of motion of sandwich shells are derived from the balance of: (1) the power of the contact forces/couples; (2) the rate of the kinetic energy; and (3) the power of the assigned forces/couples. These equations of motion are entirely expressed in term of stress resultant and stress couples. The 12 principal unknowns, which describe the motion of the sandwich shell, are: three displacements of the neutral surface of the reference layer and three rotations for each of the three layers.

The resulting fully-nonlinear equations of motion are subsequently linearized, especially for symmetric sandwich shells. Further reduction of the linearized equations, by assuming equal outer layer rotations and by assuming that the sandwich shell surface is initially flat, leads to the results obtained in Yu (1959) for the small deformable symmetric sandwich plates. The present theory is more general in that we deal with general sandwich shells, with no *a priori* assumptions in the symmetry of the sandwich shell and non zero in-plane motion as employed in Yu (1959).

It is noted that we are not following the approach adopted in Green and Naghdi (1982) for small deformation theory of laminated composite plates, where the authors merely combined the layers, with each layer described by a theory for single-layer plates that they first developed.

2. KINEMATICS OF DEFORMATION

In this section, we consider the basic kinematics for the geometrically-exact sandwich shell, and derive some important results to be used in the following chapters.

2.1. *Basic kinematic assumptions and configurations*

Show in Fig. 1 are the profile of a sandwich shell in the material configuration and the definition of the relevant geometric quantities that are crucial in the present theory. The neutral surface [to be defined in eqn (31)] of the reference layer (0) in the material configuration is coordinated by the coordinates (ξ^1, ξ^2). The transverse fiber orthogonal to the reference surface, in the material configuration is coordinated by ξ^3. Let $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ be a set of orthonormal basis vectors associated with the Cartesian coordinate (ξ^1, ξ^2, ξ^3). The neutral surface of layer (ℓ) with $\ell = \pm 1$ is at the distance $_{(\ell)}Z \geqslant 0$ from the neutral surface of the reference layer (0) with $_{(0)}Z = 0$ and is at the distance $_{(\ell)}h^+$ from the top of layer (ℓ), and at $_{(\ell)}h^-$ from the bottom of layer (ℓ). Note that $_{(\ell)}Z$ is not the ordinate, but the distance, and is thus a positive number, for $\ell = -1, 1$. The thickness of layer (ℓ) is given by

$$_{(\ell)}H := {}_{(\ell)}h^{+} + {}_{(\ell)}h^{-}, \tag{1}$$

with $_{(\ell)}h^{+} \neq {}_{(\ell)}h^{-}$ in general. The domain of the cross section of layer (ℓ) can be written as

$$\begin{aligned} {}_{(0)}\mathscr{H} &:= [-{}_{(0)}h^{-}, {}_{(0)}h^{+}], \\ {}_{(1)}\mathscr{H} &:= [({}_{(1)}Z - {}_{(1)}h^{-}), ({}_{(1)}Z + {}_{(1)}h^{-})], \\ {}_{(-1)}\mathscr{H} &:= [(-{}_{(-1)}Z - {}_{(-1)}h^{-}), (-{}_{(-1)}Z + {}_{(-1)}h^{+})], \end{aligned} \tag{2}$$

Remark 2.1. The numbering of the three layers here is $-1, 0, 1$, and is different from that employed in Vu-Quoc and Deng (1995c) and Vu-Quoc and Ebcioğlu (1995), which was $1, 2, 3$. ■

Let $\mathscr{A}$ designate the material surface of the shell and $\mathscr{H} := \bigcup_{\ell=-1}^{1} {}_{(\ell)}\mathscr{H}$ the domain of the sandwich shell in the through-the-thickness direction; then the material domain $\mathscr{B}$ of the sandwich shell can be expressed as

$$\mathscr{B} := \mathscr{A} \times \mathscr{H}, \tag{3}$$

and the material domain $_{(\ell)}\mathscr{B}$ of layer (ℓ) as

$$_{(\ell)}\mathscr{B} := \mathscr{A} \times {}_{(\ell)}\mathscr{H}. \tag{4}$$

The boundary of $\mathscr{A}$ is denoted by $\partial\mathscr{A}$. The boundary of the surface of layer (ℓ) in the material configuration is denoted by

$$_{(\ell)}\partial\mathscr{A} := \partial\mathscr{A} \times {}_{(\ell)}\mathscr{H}. \tag{5}$$

Remark 2.2. Since we are focusing on the development of the equations of motion, we will henceforth consider the case where each layer may have different material properties and different thickness, but all layers must have the same surface area in the material configuration. ■

To simplify the presentation, we use the notation $\boldsymbol{\xi} := \{\boldsymbol{\xi}^{\alpha}, \xi^{3}\} \in \mathscr{B}$ to denote the coordinates of a material point, where $\boldsymbol{\xi}^{\alpha} := \{\xi^{1}, \xi^{2}\} \in \mathscr{A}$ is referred to as the material surface coordinates, and $\xi^{3} \in \mathscr{H}$ the material through-the-thickness coordinate.†

In the development of the inextensible single-director‡ sandwich shell theory, the unit sphere, denoted by S^2, plays a central role. We set

$$S^2 := \{\mathbf{t} \in \mathscr{R}^3 \mid \|\mathbf{t}\| = 1\}. \tag{6}$$

The deformation maps that map the material configuration of each of the shell layers into the corresponding current configuration, as shown in Fig. 2, are written as §

$$\begin{aligned} {}_{(0)}\boldsymbol{\Phi}(\boldsymbol{\xi}, t) &:= {}_{(0)}\boldsymbol{\varphi}(\boldsymbol{\xi}^{\alpha}, t) + \xi^{3}\,{}_{(0)}\mathbf{t}(\boldsymbol{\xi}^{\alpha}, t), \\ {}_{(1)}\boldsymbol{\Phi}(\boldsymbol{\xi}, t) &:= {}_{(1)}\boldsymbol{\varphi}(\boldsymbol{\xi}^{\alpha}, t) + (\xi^{3} - {}_{(1)}Z)\,{}_{(1)}\mathbf{t}(\boldsymbol{\xi}^{\alpha}, t), \\ {}_{(-1)}\boldsymbol{\Phi}(\boldsymbol{\xi}, t) &:= {}_{(-1)}\boldsymbol{\varphi}(\boldsymbol{\xi}^{\alpha}, t) + (\xi^{3} + {}_{(-1)}Z)\,{}_{(-1)}\mathbf{t}(\boldsymbol{\xi}^{\alpha}, t), \end{aligned} \tag{7}$$

where $_{(\ell)}\boldsymbol{\varphi} : \mathscr{A} \mapsto \mathscr{R}^3$ is the deformation map of the neutral surface of layer (ℓ) of the

† The usual convention that Greek indices take values in $\{1, 2\}$ and Latin indices take values in $\{1, 2, 3\}$ is adopted throughout this paper.

‡ The director does not rotate about itself, i.e. there is no drill degree of freedom (d.o.f.).

§ A more general formula is given in (Naghdi [1972, p.466]) for the single-layer shell with multiple directors.

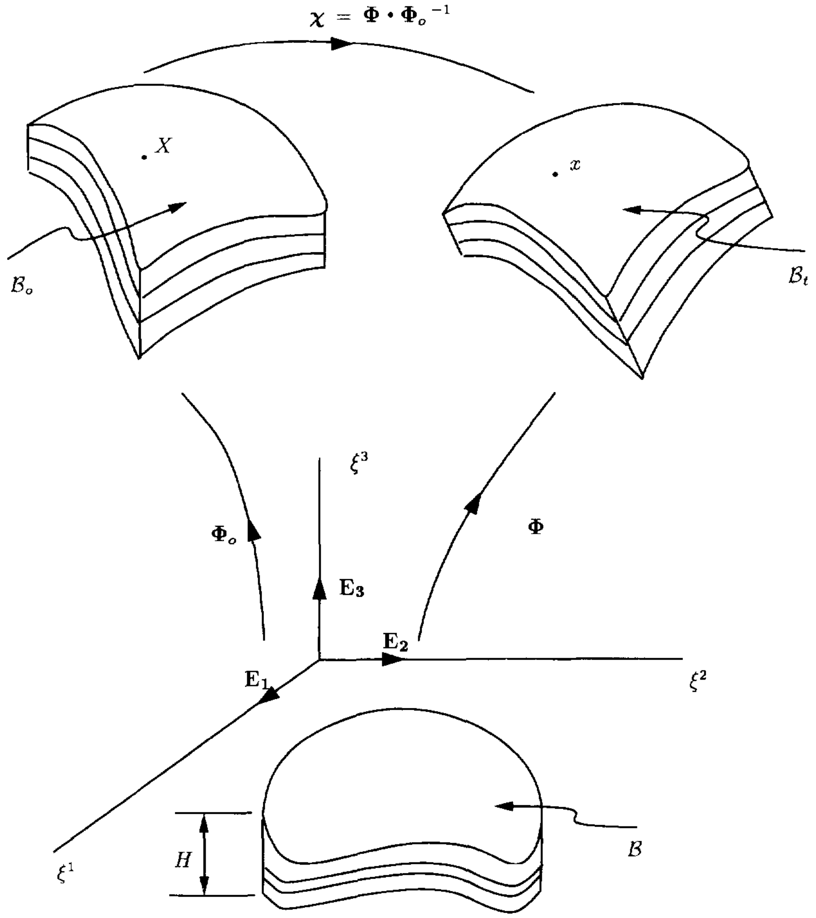

Fig. 2. Sandwich shell: material configuration $\mathscr{B}$, reference configuration $\mathscr{B}_0$, and spatial configuration $\mathscr{B}_t$.

sandwich shell to be given shortly, and ${}_{(\ell)}\mathbf{t}: \mathscr{A} \mapsto S^2$ the unit director to the neutral surface of layer (ℓ), for $\ell = -1, 0, 1$, respectively. The kinematic assumption eqn (7) embodies the following basic physical assumption: the deformation of the shell is such that points initially along the initial director ${}_{(\ell)}\mathbf{t}_0(\xi^\alpha)$ in the reference configuration remain along the director ${}_{(\ell)}\mathbf{t}(\xi^\alpha, t)$ in the spatial configuration after deformation. The fact that the director field ${}_{(\ell)}\mathbf{t}$ is straight, but not necessarily orthogonal to the deformed neutral surface of layer (ℓ), enables us to account for the shear deformation in each layer.

In particular, the deformation maps from the material configuration to the reference configuration at time $t = 0$ is obtained from eqn (7) as follows,

$$ {}_{(\ell)}\mathbf{\Phi}_0(\xi) := {}_{(\ell)}\mathbf{\Phi}(\xi, 0), \quad \text{for } \ell = -1, 0, 1. \tag{8}$$

We now assume that the overall transverse fiber of the deformed sandwich shell is continuous piecewise linear, and thus can be thought of as a chain of rigid links connected by revolute joints. Using this rigid-link assumption, the deformation maps of the neutral surface of the outer layers (1) and (-1) can be related to the deformation map of the neutral surface of the reference layer (0) as

$$ {}_{(1)}\boldsymbol{\varphi}(\xi^\alpha, t) := {}_{(0)}\boldsymbol{\varphi}(\xi^\alpha, t) + {}_{(0)}h^+ \, {}_{(0)}\mathbf{t} + {}_{(1)}h^- \, {}_{(1)}\mathbf{t}, \tag{9}$$

$$ {}_{(-1)}\boldsymbol{\varphi}(\xi^\alpha, t) := {}_{(0)}\boldsymbol{\varphi}(\xi^\alpha, t) - {}_{(0)}h^- \, {}_{(0)}\mathbf{t} - {}_{(-1)}h^+ \, {}_{(-1)}\mathbf{t}. \tag{10}$$

Remark 2.3. The deformation map for the neutral surface ${}_{(\ell)}\boldsymbol{\varphi}$ and the director field ${}_{(\ell)}\mathbf{t}$ are functions of the material surface coordinate ξ^α and the time t only, not functions of the through-the-thickness variable ξ^3. ■

Remark 2.4. By virtue of eqns (7), (9) and (10), we note that the continuation of displacements across the layer interfaces is exactly enforced. ■

The map from the neutral surface of layer (ℓ) in the material configuration to the neutral surface of layer (ℓ) in the reference configuration is defined as

$$_{(\ell)}\boldsymbol{\varphi}_0(\xi^\alpha) := {}_{(\ell)}\boldsymbol{\varphi}(\xi^\alpha, 0), \quad \text{for } \ell = -1, 0, 1. \tag{11}$$

The reader is referred to Fig. 2 for the profile of a sandwich shell in the material, reference and spatial configuration and the related deformation maps.

The deformation map for the material neutral surface of the reference layer (0) can be written as

$$_{(0)}\boldsymbol{\varphi}(\xi^\alpha, t) := \xi^\alpha \mathbf{E}_\alpha + \tilde{\mathbf{u}}(\xi^\alpha, t), \tag{12}$$

where $\tilde{\mathbf{u}}$ is the displacement of the neutral surface from the material configuration to the spatial configuration. The deformation map of the neutral surface of layer (0) from the material configuration to the reference configuration is obtained from eqn (12), by restricting the time variable to zero as follows

$$_{(0)}\boldsymbol{\varphi}_0(\xi^\alpha, 0) := \xi^\alpha \mathbf{E}_\alpha + \tilde{\mathbf{u}}_0(\xi^\alpha), \tag{13}$$

where $\tilde{\mathbf{u}}_0(\xi^\alpha) := \tilde{\mathbf{u}}(\xi^\alpha, 0)$ is the displacement from the material configuration to the reference configuration. The displacement from the reference configuration to the spatial configuration is

$$\mathbf{u} = \tilde{\mathbf{u}} - \tilde{\mathbf{u}}_0. \tag{14}$$

The principal unknowns in the present geometrically-exact sandwich shell theory are: (i) the 3 components of the displacement $\mathbf{u}$; (ii) the three angles determining the direction of the director $_{(\ell)}\mathbf{t}$ of layer (ℓ), for $\ell = -1, 0, 1$. There are thus 12 unknown kinematic quantities describing the motion of the shell. However, the inextensibility of the director $_{(\ell)}\mathbf{t}$, for $\ell = -1, 0, 1$ (which is equivalent to three constraint equations relating the 12 unknowns) reduces the total number of unknowns from 12 to nine. We will derive the governing equations of motion relating the 12 unconstrained kinematic unknowns in Section 3.

2.2. *Important kinematic quantities*

For the derivation of the equations of motion in Section 3, the following derivatives of the deformation map $_{(\ell)}\boldsymbol{\varphi}$, of layer (ℓ), are needed. The partial derivatives of $_{(\ell)}\boldsymbol{\varphi}$ with respect to the material coordinate ξ^α are as follows

$$\begin{aligned}
_{(0)}\boldsymbol{\varphi}_{,\alpha} &= \mathbf{E}_\alpha + \tilde{\mathbf{u}}_{,\alpha}, \\
_{(1)}\boldsymbol{\varphi}_{,\alpha} &= {}_{(0)}\boldsymbol{\varphi}_\alpha + {}_{(0)}h^+ \, {}_{(0)}\mathbf{t}_{,\alpha} + {}_{(1)}h^- \, {}_{(1)}\mathbf{t}_{,\alpha}, \\
_{(-1)}\boldsymbol{\varphi}_{,\alpha} &= {}_{(0)}\boldsymbol{\varphi}_\alpha - {}_{(0)}h^- \, {}_{(0)}\mathbf{t}_{,\alpha} - {}_{(-1)}h^+ \, {}_{(-1)}\mathbf{t}_{,\alpha}.
\end{aligned} \tag{15}$$

Noting that the displacement $\tilde{\mathbf{u}}_0$ of the neutral surface of layer (0) from the material configuration to the reference configuration is constant in time, and thus by virtue of eqn (14)

$$\dot{\mathbf{u}} = \dot{\bar{\mathbf{u}}}, \quad \ddot{\mathbf{u}} = \ddot{\bar{\mathbf{u}}}, \tag{16}$$

we can use the displacement $\mathbf{u}$ from the reference configuration to the spatial configuration to express the first order time rate of ${}_{(\ell)}\boldsymbol{\varphi}$ as

$$\begin{aligned} {}_{(0)}\dot{\boldsymbol{\varphi}} &= \dot{\mathbf{u}}, \\ {}_{(1)}\dot{\boldsymbol{\varphi}} &= \dot{\mathbf{u}} + {}_{(0)}h^{+}\,{}_{(0)}\dot{\mathbf{t}} + {}_{(1)}h^{-}\,{}_{(1)}\dot{\mathbf{t}}, \\ {}_{(-1)}\dot{\boldsymbol{\varphi}} &= \dot{\mathbf{u}} - {}_{(0)}h^{-}\,{}_{(0)}\dot{\mathbf{t}} - {}_{(-1)}h^{+}\,{}_{(-1)}\dot{\mathbf{t}}, \end{aligned} \tag{17}$$

the second order time rate of ${}_{(\ell)}\boldsymbol{\varphi}$ as

$$\begin{aligned} {}_{(0)}\ddot{\boldsymbol{\varphi}} &= \ddot{\mathbf{u}}, \\ {}_{(1)}\ddot{\boldsymbol{\varphi}} &= \ddot{\mathbf{u}} + {}_{(0)}h^{+}\,{}_{(0)}\ddot{\mathbf{t}} + {}_{(1)}h^{-}\,{}_{(1)}\ddot{\mathbf{t}}, \\ {}_{(-1)}\ddot{\boldsymbol{\varphi}} &= \ddot{\mathbf{u}} - {}_{(0)}h^{-}\,{}_{(0)}\ddot{\mathbf{t}} - {}_{(-1)}h^{+}\,{}_{(-1)}\ddot{\mathbf{t}}, \end{aligned} \tag{18}$$

and the first order time rate of ${}_{(\ell)}\boldsymbol{\varphi}_{,\alpha}$ as

$$\begin{aligned} {}_{(0)}\dot{\boldsymbol{\varphi}}_{,\alpha} &= \dot{\mathbf{u}}_{,\alpha}, \\ {}_{(1)}\dot{\boldsymbol{\varphi}}_{,\alpha} &= \dot{\mathbf{u}}_{,\alpha} + {}_{(0)}h^{+}\,{}_{(0)}\dot{\mathbf{t}}_{,\alpha} + {}_{(1)}h^{-}\,{}_{(1)}\dot{\mathbf{t}}_{,\alpha}, \\ {}_{(-1)}\dot{\boldsymbol{\varphi}}_{,\alpha} &= \dot{\mathbf{u}}_{,\alpha} - {}_{(0)}h^{+}\,{}_{(0)}\dot{\mathbf{t}}_{,\alpha} - {}_{(-1)}h^{-}\,{}_{(-1)}\dot{\mathbf{t}}_{,\alpha}, \end{aligned} \tag{19}$$

The spatial convected basis vectors ${}_{(\ell)}\mathbf{g}_i$, for layer (ℓ), are defined by differentiating the deformation map ${}_{(\ell)}\boldsymbol{\Phi}$ with respect to the material coordinate ξ^i:

$${}_{(\ell)}\mathbf{g}_i(\xi, t) := {}_{(\ell)}\boldsymbol{\Phi}_{,i}(\xi, t) \quad \text{for } \ell = -1, 0, 1. \tag{20}$$

Using eqn (7), we obtain

$$\begin{aligned} {}_{(0)}\mathbf{g}_\alpha(\xi, t) &= {}_{(0)}\boldsymbol{\varphi}_{,\alpha} + \xi^3\,{}_{(0)}\mathbf{t}_{,\alpha}, \\ {}_{(1)}\mathbf{g}_\alpha(\xi, t) &= {}_{(1)}\boldsymbol{\varphi}_{,\alpha} + (\xi^3 - {}_{(1)}Z)\,{}_{(1)}\mathbf{t}_{,\alpha}, \\ {}_{(-1)}\mathbf{g}_\alpha(\xi, t) &= {}_{(-1)}\boldsymbol{\varphi}_{,\alpha} + (\xi^3 + {}_{(-1)}Z)\,{}_{(-1)}\mathbf{t}_{,\alpha}, \\ {}_{(\ell)}\mathbf{g}_3(\xi, t) &= {}_{(\ell)}\mathbf{t} \quad \text{for } \ell = 1, 2, 3. \end{aligned} \tag{21}$$

The reference convected basis vectors ${}_{(\ell)}\mathbf{G}_I$ for layer (ℓ) are obtained by particularizing ${}_{(\ell)}\mathbf{g}_I$ to $t = 0$ to yield

$${}_{(\ell)}\mathbf{G}_I(\xi) := {}_{(\ell)}\mathbf{g}_I(\xi, 0) = {}_{(\ell)}\boldsymbol{\Phi}_{,I}(\xi, 0) = {}_{(\ell)}\boldsymbol{\Phi}_{0,I}(\xi). \tag{22}$$

The dual basis vectors ${}_{(\ell)}\mathbf{g}^i$ and ${}_{(\ell)}\mathbf{G}^I$ are defined by the orthogonality condition

$$\langle {}_{(\ell)}\mathbf{g}^i, {}_{(\ell)}\mathbf{g}_j \rangle = \delta^i_j \quad \text{and} \quad \langle {}_{(\ell)}\mathbf{G}^I, {}_{(\ell)}\mathbf{G}_J \rangle = \delta^I_J, \tag{23}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product, and δ^i_j and δ^I_J the Kronecker deltas.

The deformation gradient ${}_{(\ell)}\mathbf{F}$ for layer (ℓ) is, by definition, the tangent map of the deformation map ${}_{(\ell)}\chi = {}_{(\ell)}\boldsymbol{\Phi} \circ {}_{(\ell)}\boldsymbol{\Phi}_0{}^{-1}$ from the reference configurations to the current configuration, thus

$$ {}_{(\ell)}\mathbf{F} := \nabla_{(\ell)}\chi = \nabla_{(\ell)}\mathbf{\Phi} \circ [\nabla_{(\ell)}\mathbf{\Phi}_0]^{-1}, \tag{24} $$

which when expressed in the convected coordinate, is written as

$$ {}_{(\ell)}\mathbf{F} = \delta^i_{I\,(\ell)}\mathbf{g}_i \otimes {}_{(\ell)}\mathbf{G}^I. \tag{25} $$

The component form of the inverse of the deformation gradient ${}_{(\ell)}\mathbf{F}^{-1}$ for layer (ℓ) is then

$$ {}_{(\ell)}\mathbf{F}^{-1} = \delta^I_{i\,(\ell)}\mathbf{G}_I \otimes {}_{(\ell)}\mathbf{g}^i. \tag{26} $$

The time rate of the deformation gradient, which is crucial for the development of the stress power in Section 3, is given below

$$ {}_{(\ell)}\dot{\mathbf{F}} = \delta^i_{I\,(\ell)}\dot{\mathbf{g}}_i \otimes {}_{(\ell)}\mathbf{G}^I, \tag{27} $$

by virtue of eqn (25), and by the constancy of ${}_{(\ell)}\mathbf{G}^I$ with respect to time.

The Jacobian determinants of the mapping ${}_{(\ell)}\mathbf{\Phi}_0$, ${}_{(\ell)}\mathbf{\Phi}$ and ${}_{(\ell)}\chi$ are given below

$$ {}_{(\ell)}j_0 := \det[\nabla_{(\ell)}\mathbf{\Phi}_0(\xi)] = ({}_{(\ell)}\mathbf{G}_1 \times {}_{(\ell)}\mathbf{G}_2) \cdot {}_{(\ell)}\mathbf{G}_3, \tag{28} $$

$$ {}_{(\ell)}j_t := \det[\nabla_{(\ell)}\mathbf{\Phi}_0(\xi, t)] = ({}_{(\ell)}\mathbf{g}_1 \times {}_{(\ell)}\mathbf{g}_2) \cdot {}_{(\ell)}\mathbf{g}_3, \tag{29} $$

$$ {}_{(\ell)}J := \det[\nabla_{(\ell)}\chi(\xi, t)] = \frac{{}_{(\ell)}j_t}{{}_{(\ell)}j_0}, \tag{30} $$

where eqn (30) is obtained by using eqns (28) and (29) and eqn (24).

We now conclude this section by defining the neutral surface of layer (ℓ). Let ${}_{(\ell)}\rho_0$ and ${}_{(\ell)}\rho$ be the mass density in the reference and spatial configuration, $\mathscr{B}_0$ and $\mathscr{B}_t$, respectively. We select the neutral surface ${}_{(\ell)}\boldsymbol{\varphi}_0$ of layer (ℓ) such that

$$ \int_{(-1)\mathscr{H}} (\xi^3 + {}_{(-1)}Z)_{(-1)}j_{0\,(-1)}\rho_0 \,\mathrm{d}\xi^3 = \int_{(-1)\mathscr{H}} (\xi^3 + {}_{(-1)}Z)_{(-1)}j_{t\,(-1)}\rho \,\mathrm{d}\xi^3 = 0, $$

$$ \int_{(0)\mathscr{H}} \xi^3\,{}_{(0)}j_{0\,(0)}\rho_0 \,\mathrm{d}\xi^3 = \int_{(0)\mathscr{H}} \xi^3\,{}_{(0)}j_{t\,(0)}\rho \,\mathrm{d}\xi^3 = 0, $$

$$ \int_{(1)\mathscr{H}} (\xi^3 - {}_{(1)}Z)_{(1)}j_{0\,(1)}\rho_0 \,\mathrm{d}\xi^3 = \int_{(1)\mathscr{H}} (\xi^3 - {}_{(1)}Z)_{(1)}j_{t\,(1)}\rho \,\mathrm{d}\xi^3 = 0, \tag{31} $$

which does not necessarily correspond to the geometric center of the cross-section of layer (ℓ).

3. EQUATIONS OF MOTION

There are several approaches to obtain the equations of motion of geometrically-exact shells. One way is to start from the local balance of linear momentum, (i) which when integrated over the shell thickness yields the resultant balance of linear momentum, and (ii) which when integrated over the shell thickness with the weighting function ξ^3 yields the resultant balance of angular momentum, [see, e.g. Simo and Fox (1989) for the single-layer shell theory]. The symmetric condition of the Cauchy stress tensor in terms of stress resultants for the shell serves as the restriction on the constitutive laws relating the shell stress resultants. The kinematic condition of inextensibility of the director in each layer serves as additional constraint that plays a role in the weak form. In the present paper, which deals with sandwich shells, we employ a different approach to derive the equations of motion based on the principle of virtual power, as pioneered for sandwich beams in Vu-Quoc and Ebcioğlu (1995). The advantage of the adopted approach is that the weak form for the geometrically-exact sandwich shell can be readily obtained from the balance of stress power.

3.1. *Power of contact forces/couples*

The shell resultant stresses and resultant couples, and their respective conjugate strain measures can be obtained by reducing the expression of the stress power from an integration over a three-dimensional domain of the shell to an integration over the two-dimensional domain of its material surface. Let $\mathbf{P}$ be the first Piola–Kirchhoff stress tensor, then the stress power of the shell expressed with respect to the reference configuration is

$$\mathscr{P}_c := \int_{\mathscr{B}_0} \mathbf{P} \cdot\cdot \dot{\mathbf{F}} \, d\mathscr{V}_0, \tag{32}$$

where $d\mathscr{V}_0$ is the infinitesimal reference volume.

The first Piola–Kirchhoff stress tensor in layer (ℓ) can be written as [see, e.g. Malvern (1969, p. 222)]

$${}_{(\ell)}\mathbf{P} = {}_{(\ell)}J_{(\ell)}\mathbf{F}^{-1} \cdot {}_{(\ell)}\boldsymbol{\sigma}, \tag{33}$$

where ${}_{(\ell)}J$ is defined in eqn (30), ${}_{(\ell)}\mathbf{F}^{-1}$ in eqn (26), and ${}_{(\ell)}\boldsymbol{\sigma}$ is the Cauchy stress tensor in layer (ℓ). Using the expression for ${}_{(\ell)}\mathbf{F}^{-1}$ in eqn (26), the first Piola–Kirchhoff stress tensor ${}_{(\ell)}\mathbf{P} = {}_{(\ell)}P^{Ij}{}_{(\ell)}\mathbf{G}_I \otimes {}_{(\ell)}\mathbf{g}_j$ can be expressed as

$${}_{(\ell)}\mathbf{P} = {}_{(\ell)}J\delta^I_i{}_{(\ell)}\mathbf{G}_I \otimes {}_{(\ell)}\mathbf{g}^i \cdot {}_{(\ell)}\sigma^{jk}{}_{(\ell)}\mathbf{g}_j \otimes {}_{(\ell)}\mathbf{g}_k = {}_{(\ell)}\mathbf{J}\delta^I_i{}_{(\ell)}\sigma^{ij}{}_{(\ell)}\mathbf{G}_I \otimes {}_{(\ell)}\mathbf{g}_j. \tag{34}$$

With ${}_{(\ell)}\dot{\mathbf{F}}$ given in eqn (27), we obtain the following stress power per unit reference volume for layer (ℓ)

$$\begin{aligned} {}_{(\ell)}\mathbf{P} \cdot\cdot {}_{(\ell)}\dot{\mathbf{F}} &:= {}_{(\ell)}J[(\delta^I_i{}_{(\ell)}\mathbf{G}_I \otimes {}_{(\ell)}\mathbf{g}^i) \cdot {}_{(\ell)}\boldsymbol{\sigma}] \cdot\cdot [\delta^k_J{}_{(\ell)}\dot{\mathbf{g}}_k \otimes {}_{(\ell)}\mathbf{G}^J] \\ &= {}_{(\ell)}J\delta^I_i\delta^k_J({}_{(\ell)}\mathbf{G}_I \cdot {}_{(\ell)}\mathbf{G}^J){}_{(\ell)}\mathbf{g}^i \cdot {}_{(\ell)}\boldsymbol{\sigma} \cdot {}_{(\ell)}\dot{\mathbf{g}}_k \\ &= {}_{(\ell)}J_{(\ell)}\mathbf{g}^i \cdot {}_{(\ell)}\boldsymbol{\sigma} \cdot {}_{(\ell)}\dot{\mathbf{g}}_i, \end{aligned} \tag{35}$$

where ${}_{(\ell)}\mathbf{G}_I \cdot {}_{(\ell)}\mathbf{G}^J = \delta^J_I$. The stress power for the sandwich shell can now be obtained as

$$\begin{aligned} \mathscr{P}_c := \int_{\mathscr{B}_0} \mathbf{P} \cdot\cdot \dot{\mathbf{F}} \, d\mathscr{V}_0 &= \sum_{\ell=-1}^{1} \int_{{}_{(\ell)}\mathscr{B}_0} {}_{(\ell)}\mathbf{P} \cdot\cdot {}_{(\ell)}\dot{\mathbf{F}} \, d_{(\ell)}\mathscr{V}_0 \\ &= \sum_{\ell=-1}^{1} \int_{{}_{(\ell)}\mathscr{B}_0} {}_{(\ell)}J_{(\ell)}\mathbf{g}^i \cdot {}_{(\ell)}\boldsymbol{\sigma} \cdot {}_{(\ell)}\dot{\mathbf{g}}_i \, d_{(\ell)}\mathscr{V}_0, \end{aligned} \tag{36}$$

where ${}_{(\ell)}\mathscr{B}_0$ designates the reference body for layer (ℓ), and $d_{(\ell)}\mathscr{V}_0$ the infinitesimal reference volume in layer (ℓ). With the definition of ${}_{(\ell)}J$ and ${}_{(\ell)}j_t$ in eqns (30) and (29), respectively, and by the conservation of mass, we have

$${}_{(\ell)}J \, d_{(\ell)}\mathscr{V}_0 = d_{(\ell)}\mathscr{V}_t = {}_{(\ell)}j_t \, d_{(\ell)}\mathscr{V}, \tag{37}$$

where $d_{(\ell)}\mathscr{V} := d\mathscr{A} \, d\xi^3$ is the infinitesimal material volume in layer (ℓ). Using the definition (4) of the material domain ${}_{(\ell)}\mathscr{B}$ of layer (ℓ), the definition (21) of the basis vector ${}_{(\ell)}\mathbf{g}_i$, and expression (37), the stress power, eqn (36), is now expressed in the material domain as follows:

$$\mathscr{P}_c := \int_{\mathscr{A}}\int_{(1)\mathscr{H}} {}_{(1)}j_t\{({}_{(1)}\mathbf{g}^\alpha \cdot {}_{(1)}\boldsymbol{\sigma})\cdot[{}_{(1)}\dot{\boldsymbol{\varphi}}_{,\alpha} + (\xi^3 - {}_{(1)}Z){}_{(1)}\dot{\mathbf{t}}_{,\alpha}] + {}_{(1)}\mathbf{t}\cdot{}_{(1)}\boldsymbol{\sigma}\cdot{}_{(1)}\dot{\mathbf{t}}\}\,\mathrm{d}\mathscr{A}\,\mathrm{d}\xi^3$$
$$+ \int_{\mathscr{A}}\int_{(0)\mathscr{H}} {}_{(0)}j_t\{({}_{(0)}\mathbf{g}^\alpha \cdot {}_{(0)}\boldsymbol{\sigma})\cdot[{}_{(0)}\dot{\boldsymbol{\varphi}}_{,\alpha} + \xi^3\,{}_{(0)}\dot{\mathbf{t}}_{,\alpha}] + {}_{(0)}\mathbf{t}\cdot{}_{(0)}\boldsymbol{\sigma}\cdot{}_{(0)}\dot{\mathbf{t}}\}\,\mathrm{d}\mathscr{A}\,\mathrm{d}\xi^3$$
$$+ \int_{\mathscr{A}}\int_{(-1)\mathscr{H}} {}_{(-1)}j_t\{({}_{(-1)}\mathbf{g}^\alpha \cdot {}_{(-1)}\boldsymbol{\sigma})\cdot[{}_{(-1)}\dot{\boldsymbol{\varphi}}_{,\alpha} + (\xi^3 + {}_{(-1)}Z){}_{(-1)}\dot{\mathbf{t}}_{,\alpha}]$$
$$+ {}_{(-1)}\mathbf{t}\cdot{}_{(-1)}\boldsymbol{\sigma}\cdot{}_{(-1)}\dot{\mathbf{t}}\}\,\mathrm{d}\mathscr{A}\,\mathrm{d}\xi^3. \tag{38}$$

Upon defining the resultant forces ${}_{(\ell)}\mathbf{n}^\alpha$, the resultant couples ${}_{(\ell)}\mathbf{m}^\alpha$ and the resultant director couple ${}_{(\ell)}\ell$ for layer (ℓ) are, respectively,

$$ {}_{(\ell)}\mathbf{n}^\alpha := \int_{(\ell)\mathscr{H}} {}_{(\ell)}j_t\,{}_{(\ell)}\mathbf{g}^\alpha\cdot{}_{(\ell)}\boldsymbol{\sigma}\,\mathrm{d}\xi^3, \quad \text{for } \ell = -1, 0, 1, \tag{39}$$

$$ {}_{(0)}\tilde{\mathbf{m}}^\alpha := \int_{(0)\mathscr{H}} {}_{(0)}j_t\,\xi^3\,{}_{(0)}\mathbf{g}^\alpha\cdot{}_{(0)}\boldsymbol{\sigma}\,\mathrm{d}\xi^3,$$
$$ {}_{(1)}\tilde{\mathbf{m}}^\alpha := \int_{(1)\mathscr{H}} {}_{(1)}j_t(\xi^3 - {}_{(1)}Z){}_{(1)}\mathbf{g}^\alpha\cdot{}_{(1)}\boldsymbol{\sigma}\,\mathrm{d}\xi^3,$$
$$ {}_{(-1)}\tilde{\mathbf{m}}^\alpha := \int_{(-1)\mathscr{H}} {}_{(-1)}j_t(\xi^3 + {}_{(-1)}Z){}_{(-1)}\mathbf{g}^\alpha\cdot{}_{(-1)}\boldsymbol{\sigma}\,\mathrm{d}\xi^3, \tag{40}$$

$$ {}_{(\ell)}\ell := \int_{(\ell)\mathscr{H}} {}_{(\ell)}j_t\,{}_{(\ell)}\mathbf{g}^3\cdot{}_{(\ell)}\boldsymbol{\sigma}\,\mathrm{d}\xi^3, \quad \text{for } \ell = -1, 0, 1; \tag{41}$$

the stress power, eqn (38), can be simplified as

$$\mathscr{P}_c = \sum_{\ell=-1}^{1}\int_{\mathscr{A}} [{}_{(\ell)}\mathbf{n}^\alpha\cdot{}_{(\ell)}\dot{\boldsymbol{\varphi}}_{,\alpha} + {}_{(\ell)}\tilde{\mathbf{m}}^\alpha\cdot{}_{(\ell)}\dot{\mathbf{t}}_{,\alpha} + {}_{(\ell)}\ell\cdot{}_{(\ell)}\dot{\mathbf{t}}]\,\mathrm{d}\mathscr{A}. \tag{42}$$

Remark 3.1. Our definitions for the resultant forces/couples and the director couples are different from those given in Simo and Fox (1989) by a factor $1/{}_{(\ell)}j_t$, but are the same as the weighted surface tensors ($\mathbf{N}^{\alpha\beta}$, $\mathbf{M}^{\alpha\beta}$, and $\mathbf{Q}^\alpha$) given in Green and Zerna (1968, p. 376) except for the director couple ${}_{(\ell)}\ell$. We note that the definitions employed in Simo and Fox (1989) correspond to the physical resultant forces/couples as explained in Green and Zerna (1968, p. 377). We employ here the weighted surface tensors rather than the physical resultants as in Simo & Fox (1989), because the weighted surface tensors significantly simplify the derivation of the equations of motion. The reason we call ${}_{(\ell)}\ell$ the resultant director couple, rather than "through-the-thickness resultant force" as in Simo and Fox (1989), will be explained later in Remark 3.4. ■

In order to obtain the equations of motion of the sandwich shell from the balance of power, we reorganize the contact stress power, eqn (42) with the rate of the displacement $\dot{\mathbf{u}}$ and the rate of the director ${}_{(\ell)}\dot{\mathbf{t}}$, as the common factors.

Consider the first term in eqn (42); using expression (19) for ${}_{(\ell)}\dot{\boldsymbol{\varphi}}_{,\alpha}$, we first gather all the terms with factor ${}_{(0)}\dot{\boldsymbol{\varphi}}_{,\alpha} = \dot{\mathbf{u}}_{,\alpha}$

$$\int_{\mathscr{A}}\sum_{\ell=-1}^{1} {}_{(\ell)}\mathbf{n}^\alpha\cdot\dot{\mathbf{u}}_{,\alpha}\,\mathrm{d}\mathscr{A} = \int_{\mathscr{A}} \hat{\mathbf{n}}^\alpha\cdot\dot{\mathbf{u}}_{,\alpha}\,\mathrm{d}\mathscr{A}, \tag{43}$$

where

$$\hat{\mathbf{n}}^\alpha := \sum_{\ell=-1}^{1} {}_{(\ell)}\mathbf{n}^\alpha. \tag{44}$$

Integrating eqn (43) by parts yields

$$\int_{\mathscr{A}} \sum_{\ell=-1}^{1} {}_{(\ell)}\mathbf{n}^\alpha \cdot \dot{\mathbf{u}}_{,\alpha}\, \mathrm{d}\mathscr{A} = \sum_{\ell=-1}^{1} \oint_{\partial\mathscr{A}} {}_{(\ell)}\mathbf{n}^\alpha \cdot \dot{\mathbf{u}}_{(\ell)}\nu_\alpha\, \mathrm{d}(\partial\mathscr{A}) - \int_{\mathscr{A}} \hat{\mathbf{n}}^\alpha{}_{,\alpha} \cdot \dot{\mathbf{u}}\, \mathrm{d}\mathscr{A}, \tag{45}$$

where $\dot{\mathbf{u}}$ is the only common vector, and where $\mathrm{d}(\partial\mathscr{A})$ is the boundary of the material surface $\mathrm{d}\mathscr{A}$, and ${}_{(\ell)}\nu_\alpha$ is the component of the normal ${}_{(\ell)}\boldsymbol{\nu}$ at the boundary of the neutral surface layer (ℓ) such that

$$_{(\ell)}\boldsymbol{\nu} = {}_{(\ell)}\nu_\alpha \mathbf{E}^\alpha. \tag{46}$$

Remark 3.2. Equation (45) is obtained on the assumption that in the material configuration, we have

$$_{(\ell)}\nu_\alpha = {}_{(0)}\nu_\alpha, \quad \forall\alpha = 1,2, \quad \forall\ell = -1, 1. \tag{47}$$

This assumption means that the shell in the material configuration is like a straight cylinder. The role of the notation ${}_{(\ell)}\nu_\alpha$ in eqn (45) is simply for the symmetry of the formula. Generally, in the reference (initial, stress free) configuration,

$$_{(\ell)}\nu_{0\alpha} \neq {}_{(0)}\nu_{0\alpha}, \quad \forall\alpha = 1,2, \quad \forall\ell = -1,1. \quad \blacksquare \tag{48}$$

Next, we gather all terms having ${}_{(0)}\dot{\mathbf{t}}$ or ${}_{(0)}\dot{\mathbf{t}}_{,\alpha}$ as common factors, coming from the use of eqns $(19)_2$, $(19)_3$ in the first term of eqn (42), and also coming from the second and third terms of eqn (42) as follows:

$$\int_{\mathscr{A}} [({}_{(0)}\tilde{\mathbf{m}}^\alpha \cdot {}_{(0)}\dot{\mathbf{t}}_{,\alpha} + {}_{(0)}\boldsymbol{\ell} \cdot {}_{(0)}\dot{\mathbf{t}}) + ({}_{(0)}h^+\,{}_{(1)}\mathbf{n}^\alpha - {}_{(0)}h^-\,{}_{(-1)}\mathbf{n}^\alpha) \cdot {}_{(0)}\dot{\mathbf{t}}_{,\alpha}]\, \mathrm{d}\mathscr{A}. \tag{49}$$

Integrating eqn (49) over the layer neutral surface by parts, we obtain an expression in which ${}_{(0)}\dot{\mathbf{t}}$ is the only common factor:

$$\begin{aligned} &\int_{\mathscr{A}} [({}_{(0)}\tilde{\mathbf{m}}^\alpha \cdot {}_{(0)}\dot{\mathbf{t}}_{,\alpha} + {}_{(0)}\boldsymbol{\ell} \cdot {}_{(0)}\dot{\mathbf{t}}) + ({}_{(0)}h^+\,{}_{(1)}\mathbf{n}^\alpha - {}_{(0)}h^-\,{}_{(-1)}\mathbf{n}^\alpha) \cdot {}_{(0)}\dot{\mathbf{t}}_{,\alpha}]\, \mathrm{d}\mathscr{A} \\ &\quad = \oint_{\partial\mathscr{A}} ({}_{(0)}\tilde{\mathbf{m}}^\alpha + {}_{(0)}h^+\,{}_{(1)}\mathbf{n}^\alpha - {}_{(0)}h^-\,{}_{(-1)}\mathbf{n}^\alpha) \cdot {}_{(0)}\dot{\mathbf{t}}_{(0)}\nu_\alpha\, \mathrm{d}(\partial\mathscr{A}) \\ &\qquad - \int_{\mathscr{A}} [({}_{(0)}\tilde{\mathbf{m}}^\alpha + {}_{(0)}h^+\,{}_{(1)}\mathbf{n}^\alpha - {}_{(0)}h^-\,{}_{(-1)}\mathbf{n}^\alpha)_{,\alpha} - {}_{(0)}\boldsymbol{\ell}] \cdot {}_{(0)}\dot{\mathbf{t}}\, \mathrm{d}\mathscr{A}. \end{aligned} \tag{50}$$

Next, concerning the top layer (1), we gather all the terms in eqn (42) (after substituting eqn $(19)_2$) that have ${}_{(1)}\dot{\mathbf{t}}$ or ${}_{(1)}\dot{\mathbf{t}}_{,\alpha}$ as common factors to have

$$\int_{\mathscr{A}} ({}_{(1)}\tilde{\mathbf{m}}^\alpha \cdot {}_{(1)}\dot{\mathbf{t}}_{,\alpha} + {}_{(1)}\boldsymbol{\ell} \cdot {}_{(1)}\dot{\mathbf{t}} + {}_{(1)}h^-\,{}_{(1)}\mathbf{n}^\alpha \cdot {}_{(1)}\dot{\mathbf{t}}_{,\alpha})\, \mathrm{d}\mathscr{A}, \tag{51}$$

which after an integration by parts yields an expression having ${}_{(1)}\dot{\mathbf{t}}$ as the only common factor:

$$\int_{\mathscr{A}} ({}_{(1)}\tilde{\mathbf{m}}^{\alpha} \cdot {}_{(1)}\dot{\mathbf{t}}_{,\alpha} + {}_{(1)}\ell \cdot {}_{(1)}\dot{\mathbf{t}} + {}_{(1)}h^{-}\,{}_{(1)}\mathbf{n}^{\alpha} \cdot {}_{(1)}\dot{\mathbf{t}}_{,\alpha})\, \mathrm{d}\mathscr{A} = \oint_{\partial\mathscr{A}} ({}_{(1)}\tilde{\mathbf{m}}^{\alpha} + {}_{(1)}h^{-}\,{}_{(1)}\mathbf{n}^{\alpha}) \cdot {}_{(1)}\dot{\mathbf{t}}_{(1)}\nu_{\alpha}\, \mathrm{d}(\partial\mathscr{A})$$
$$- \int_{\mathscr{A}} [({}_{(1)}\tilde{\mathbf{m}}^{\alpha} + {}_{(1)}h^{-}\,{}_{(1)}\mathbf{n}^{\alpha})_{,\alpha} - {}_{(1)}\ell] \cdot {}_{(1)}\dot{\mathbf{t}}\, \mathrm{d}\mathscr{A}. \quad (52)$$

Finally, concerning the bottom layer (-1), we gather all the terms in eqn (42) (after substituting eqn $(19)_3$) having ${}_{(-1)}\dot{\mathbf{t}}$ or ${}_{(-1)}\dot{\mathbf{t}}_{,\alpha}$ as common factors to have

$$\int_{\mathscr{A}} ({}_{(-1)}\tilde{\mathbf{m}}^{\alpha} \cdot {}_{(-1)}\dot{\mathbf{t}}_{,\alpha} + {}_{(-1)}\ell \cdot {}_{(-1)}\dot{\mathbf{t}} - {}_{(-1)}h^{+}\,{}_{(-1)}\mathbf{n}^{\alpha} \cdot {}_{(-1)}\dot{\mathbf{t}}_{,\alpha})\, \mathrm{d}\mathscr{A}, \quad (53)$$

which after an integration by parts yields an expression having ${}_{(-1)}\dot{\mathbf{t}}$ as the only common factor

$$\int_{\mathscr{A}} ({}_{(-1)}\tilde{\mathbf{m}}^{\alpha} \cdot {}_{(-1)}\dot{\mathbf{t}}_{,\alpha} + {}_{(-1)}\ell \cdot {}_{(-1)}\dot{\mathbf{t}} - {}_{(-1)}h^{+}\,{}_{(-1)}\mathbf{n}^{\alpha} \cdot {}_{(-1)}\dot{\mathbf{t}}_{,\alpha})\, \mathrm{d}\mathscr{A}$$
$$= \oint_{\partial\mathscr{A}} ({}_{(-1)}\tilde{\mathbf{m}}^{\alpha} - {}_{(-1)}h^{+}\,{}_{(-1)}\mathbf{n}^{\alpha}) \cdot {}_{(-1)}\dot{\mathbf{t}}_{(-1)}\nu_{\alpha}\, \mathrm{d}(\partial\mathscr{A})$$
$$- \int_{\mathscr{A}} [({}_{(-1)}\tilde{\mathbf{m}}^{\alpha} - {}_{(-1)}h^{+}\,{}_{(-1)}\mathbf{n}^{\alpha})_{,\alpha} - {}_{(-1)}\ell] \cdot {}_{(-1)}\dot{\mathbf{t}}\, \mathrm{d}\mathscr{A}. \quad (54)$$

Using eqns (45), (50), (52) and (54) the total contact stress power, eqn (42), can be written as

$$\mathscr{P}_{c} = \int_{\mathscr{B}_0} \mathbf{P} \cdot\cdot \dot{\mathbf{F}}\, \mathrm{d}\mathscr{V}_0 = \sum_{\ell=-1}^{1} \int_{\mathscr{A}} [{}_{\ell}\mathbf{n}^{\alpha} \cdot {}_{\ell}\dot{\boldsymbol{\varphi}}_{,\alpha} + {}_{(\ell)}\tilde{\mathbf{m}}^{\alpha} \cdot {}_{(\ell)}\dot{\mathbf{t}}_{,\alpha} + {}_{(\ell)}\ell \cdot {}_{(\ell)}\dot{\mathbf{t}}]\, \mathrm{d}\mathscr{A}$$
$$= -\int_{\mathscr{A}} \hat{\mathbf{n}}^{\alpha}{}_{,\alpha} \cdot \dot{\mathbf{u}}\, \mathrm{d}\mathscr{A} - \int_{\mathscr{A}} [({}_{(1)}\tilde{\mathbf{m}}^{\alpha} + {}_{(1)}h^{-}\,{}_{(1)}\mathbf{n}^{\alpha})_{,\alpha} - {}_{(1)}\ell] \cdot {}_{(1)}\dot{\mathbf{t}}\, \mathrm{d}\mathscr{A}$$
$$- \int_{\mathscr{A}} [({}_{(0)}\tilde{\mathbf{m}}^{\alpha} + {}_{(0)}h^{+}\,{}_{(1)}\mathbf{n}^{\alpha} - {}_{(0)}h^{-}\,{}_{(-1)}\mathbf{n}^{\alpha})_{,\alpha} - {}_{(0)}\ell] \cdot {}_{(0)}\dot{\mathbf{t}}\, \mathrm{d}\mathscr{A}$$
$$- \int_{\mathscr{A}} [({}_{(-1)}\tilde{\mathbf{m}}^{\alpha} - {}_{(-1)}h^{+}\,{}_{(-1)}\mathbf{n}^{\alpha})_{,\alpha} - {}_{(-1)}\ell] \cdot {}_{(-1)}\dot{\mathbf{t}}\, \mathrm{d}\mathscr{A}$$
$$+ \sum_{\ell=-1}^{1} \oint_{\partial\mathscr{A}} {}_{(\ell)}\mathbf{n}^{\alpha} \cdot \dot{\mathbf{u}}_{(\ell)}\nu_{\alpha}\, \mathrm{d}(\partial\mathscr{A}) + \oint_{\partial\mathscr{A}} ({}_{(1)}\tilde{\mathbf{m}}^{\alpha} + {}_{(1)}h^{-}\,{}_{(1)}\mathbf{n}^{\alpha}) \cdot {}_{(1)}\dot{\mathbf{t}}_{(1)}\nu_{\alpha}\, \mathrm{d}(\partial\mathscr{A})$$
$$+ \oint_{\partial\mathscr{A}} ({}_{(0)}\tilde{\mathbf{m}}^{\alpha} + {}_{(0)}h^{+}\,{}_{(1)}\mathbf{n}^{\alpha} - {}_{(0)}h^{-}\,{}_{(-1)}\mathbf{n}^{\alpha}) \cdot {}_{(0)}\dot{\mathbf{t}}_{(0)}\nu_{\alpha}\, \mathrm{d}(\partial\mathscr{A})$$
$$+ \oint_{\partial\mathscr{A}} ({}_{(-1)}\tilde{\mathbf{m}}^{\alpha} - {}_{(-1)}h^{+}\,{}_{(-1)}\mathbf{n}^{\alpha}) \cdot {}_{(-1)}\dot{\mathbf{t}}_{(-1)}\nu_{\alpha}\, \mathrm{d}(\partial\mathscr{A}) \quad (55)$$

3.2. *Power of assigned forces/couples*

Let $\mathbf{n}^*$ denote the distributed assigned force on the neutral surface of the reference layer (0) and ${}_{(\ell)}\tilde{\mathbf{m}}^*$ the distributed assigned moment on layer (ℓ). Also let $\mathbf{n}^{*\alpha}$ denote the assigned force on the facet with normal ${}_{(0)}\mathbf{g}^{\alpha}$ on the boundary $\mathrm{d}(\partial\mathscr{A})$ (see the definition of ${}_{(\ell)}\mathbf{n}^{\alpha}$ in eqn (39)), and ${}_{(\ell)}\tilde{\mathbf{m}}^{*\alpha}$ the assigned couples on the facet with normal ${}_{(\ell)}\mathbf{g}^{\alpha}$ on the boundary ${}_{(\ell)}\mathrm{d}(\partial\mathscr{A}) := \mathrm{d}(\partial\mathscr{A}) \times {}_{(\ell)}\mathscr{H}$ of the neutral surface of layer (ℓ). The power of the assigned forces and couples is written as follows

$$\mathscr{P}_{\mathrm{a}} = \int_{\mathscr{A}} \left(\mathbf{n}^* \cdot \dot{\mathbf{u}} + \sum_{\ell=-1}^{1} {}_{(\ell)}\tilde{\mathbf{m}}^* \cdot {}_{(\ell)}\dot{\mathbf{t}} \right) \mathrm{d}\mathscr{A} + \oint_{\partial\mathscr{A}} \left(\mathbf{n}^{*\alpha} \cdot \dot{\mathbf{u}}_{(0)} v_\alpha + \sum_{\ell=-1}^{1} {}_{(\ell)}\tilde{\mathbf{m}}^{*\alpha} \cdot {}_{(\ell)}\dot{\mathbf{t}}_{(\ell)} v_\alpha \right) \mathrm{d}(\partial\mathscr{A}). \tag{56}$$

3.3. *Rate of kinetic energy*

Let ${}_{(\ell)}\rho$ denote the mass density of layer (ℓ) in the spatial configuration. The kinetic energy $\mathscr{K}$ of the sandwich shell is given by

$$\mathscr{K} = \frac{1}{2} \sum_{\ell=-1}^{1} \int_{{}_{(\ell)}\mathscr{B}_t} {}_{(\ell)}\rho_{(\ell)}\dot{\mathbf{\Phi}} \cdot {}_{(\ell)}\dot{\mathbf{\Phi}} \, \mathrm{d}({}_{(\ell)}\mathscr{B}_t), \tag{57}$$

where ${}_{(\ell)}\mathscr{B}_t = {}_{(\ell)}\mathbf{\Phi}({}_{(\ell)}\mathscr{B})$ is the domain of layer (ℓ) in the current configuration.

By the Reynolds transport theorem and the conservation of mass [see, e.g. Malvern (1969, p. 210)], we obtain the material time derivative of the kinetic energy $\mathscr{K}$ in eqn (57) as

$$\frac{\mathrm{d}\mathscr{K}}{\mathrm{d}t} = \sum_{\ell=-1}^{1} \int_{{}_{(\ell)}\mathscr{B}_t} {}_{(\ell)}\rho_{(\ell)}\ddot{\mathbf{\Phi}} \cdot {}_{(\ell)}\dot{\mathbf{\Phi}} \, \mathrm{d}({}_{(\ell)}\mathscr{B}_t), \tag{58}$$

which when expressed in the material configuration is given by

$$\frac{\mathrm{d}\mathscr{K}}{\mathrm{d}t} = \sum_{\ell=-1}^{1} \int_{\mathscr{A}} \int_{{}_{(\ell)}\mathscr{H}} {}_{(\ell)}\rho_{(\ell)} j_{(\ell)}\ddot{\mathbf{\Phi}} \cdot {}_{(\ell)}\dot{\mathbf{\Phi}} \, \mathrm{d}\mathscr{A} \, \mathrm{d}\xi^3. \tag{59}$$

To simplify the derivation, we introduce the following notations

$${}_{(0)}\boldsymbol{\psi} := {}_{(0)}\boldsymbol{\varphi} = \xi^\alpha \mathbf{E}_\alpha + \tilde{\mathbf{u}}(\xi^\alpha, t), \tag{60}$$

$${}_{(1)}\boldsymbol{\psi} := {}_{(0)}\boldsymbol{\varphi} + {}_{(0)}h^+ ({}_{(0)}\mathbf{t} - {}_{(1)}\mathbf{t}), \tag{61}$$

$${}_{(-1)}\boldsymbol{\psi} := {}_{(0)}\boldsymbol{\varphi} - {}_{(0)}h^- ({}_{(0)}\mathbf{t} - {}_{(-1)}\mathbf{t}). \tag{62}$$

Note that ${}_{(\ell)}\boldsymbol{\psi}$ is not a function of the through-the-thickness coordinate ξ^3, and is introduced to isolate the terms with ξ^3 as a factor from the rest as follows.

$${}_{(\ell)}\mathbf{\Phi} = {}_{(\ell)}\boldsymbol{\psi} + \xi^3 \, {}_{(\ell)}\mathbf{t}, \quad \text{for } \ell = -1, 0, 1. \tag{63}$$

The term ${}_{(\ell)}\boldsymbol{\psi}$ will play a special role in the integration through the thickness in the following derivation. Define

$${}_{(\ell)}A^k_\rho := \int_{{}_{(\ell)}\mathscr{H}} {}_{(\ell)}\rho_{(\ell)} j_t (\xi^3)^k \, \mathrm{d}\xi^3, \quad \text{for} \begin{cases} k = 0, 1, 2, \\ \ell = -1, 0, 1, \end{cases} \tag{64}$$

where ${}_{(\ell)}A^0_\rho$ is the mass per unit underformed area of layer (ℓ), ${}_{(\ell)}A^1_\rho$ is the mass moment per unit underformed area of layer (ℓ), and ${}_{(\ell)}A^2_\rho$ the mass moment of inertia per unit underformed area of layer (ℓ). Since ${}_{(\ell)}\boldsymbol{\psi}$ is not a function of ξ^3, and can therefore be taken out of the integration over the thickness, the time derivative of the kinetic energy, eqn (59) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{K} = \sum_{\ell=-1}^{1}\int_{\mathscr{A}}\int_{(\ell)\mathscr{H}} {}_{(\ell)}\rho\, {}_{(\ell)}j_t\,({}_{(\ell)}\ddot{\boldsymbol{\psi}} + \xi^3\, {}_{(\ell)}\ddot{\mathbf{t}}) \cdot ({}_{(\ell)}\dot{\boldsymbol{\psi}} + \xi^3\, {}_{(\ell)}\dot{\mathbf{t}})\, \mathrm{d}\mathscr{A}\, \mathrm{d}\xi^3$$

$$= \sum_{\ell=-1}^{1}\int_{\mathscr{A}} \{ {}_{(\ell)}A^0_{\rho}\, {}_{(\ell)}\ddot{\boldsymbol{\psi}} \cdot {}_{(\ell)}\dot{\boldsymbol{\psi}} + {}_{(\ell)}\mathbf{A}^1_{\rho}({}_{(\ell)}\ddot{\boldsymbol{\psi}} \cdot {}_{(\ell)}\dot{\mathbf{t}} + {}_{(\ell)}\ddot{\mathbf{t}} \cdot {}_{(\ell)}\dot{\boldsymbol{\psi}})$$

$$+ {}_{(\ell)}A^2_{\rho}\, {}_{(\ell)}\ddot{\mathbf{t}} \cdot {}_{(\ell)}\dot{\mathbf{t}} \}\, \mathrm{d}\mathscr{A}$$

$$= \sum_{\ell=-1}^{1}\int_{\mathscr{A}} [{}_{(\ell)}\mathbf{f} \cdot {}_{(\ell)}\dot{\boldsymbol{\psi}} + {}_{(\ell)}\mathbf{q} \cdot {}_{(\ell)}\dot{\mathbf{t}}]\, \mathrm{d}\mathscr{A}, \tag{65}$$

where ${}_{(\ell)}\mathbf{f}$ and ${}_{(\ell)}\mathbf{q}$ are the inertia force and inertia couple defined as

$${}_{(\ell)}\mathbf{f} := ({}_{(\ell)}A^0_{\rho}\, {}_{(\ell)}\ddot{\boldsymbol{\psi}} + {}_{(\ell)}A^1_{\rho}\, {}_{(\ell)}\ddot{\mathbf{t}}),$$

$${}_{(\ell)}\mathbf{q} := ({}_{(\ell)}A^1_{\rho}\, {}_{(\ell)}\ddot{\boldsymbol{\psi}} + {}_{(\ell)}A^2_{\rho}\, {}_{(\ell)}\ddot{\mathbf{t}}), \quad \text{for } \ell = -1, 0, 1. \tag{66}$$

To obtain the equations of motion for the sandwich shell, we reorganize the rate of kinetic energy according to the 12 principal unknowns (prior to the introduction of the 3 constraints of the inextensibility of the directors), as introduced in Section 2.1. From eqn (60), the first and second time derivatives of ${}_{(0)}\boldsymbol{\psi}$ can be written as

$${}_{(0)}\dot{\boldsymbol{\psi}} = {}_{(0)}\dot{\boldsymbol{\varphi}} = \dot{\mathbf{u}}, \tag{67}$$

$${}_{(0)}\ddot{\boldsymbol{\psi}} = {}_{(0)}\ddot{\boldsymbol{\varphi}} = \ddot{\mathbf{u}}, \tag{68}$$

and thus, from eqns (61) and (62) the first and second time derivatives of ${}_{(1)}\boldsymbol{\psi}$ and ${}_{(-1)}\boldsymbol{\psi}$ are

$${}_{(1)}\dot{\boldsymbol{\psi}} = \dot{\mathbf{u}} + {}_{(0)}h^+({}_{(0)}\dot{\mathbf{t}} - {}_{(1)}\dot{\mathbf{t}}), \tag{69}$$

$${}_{(-1)}\dot{\boldsymbol{\psi}} = \dot{\mathbf{u}} - {}_{(0)}h^-({}_{(0)}\dot{\mathbf{t}} - {}_{(-1)}\dot{\mathbf{t}}), \tag{70}$$

$${}_{(1)}\ddot{\boldsymbol{\psi}} = \ddot{\mathbf{u}} + {}_{(0)}h^+({}_{(0)}\ddot{\mathbf{t}} - {}_{(1)}\ddot{\mathbf{t}}), \tag{71}$$

$${}_{(-1)}\ddot{\boldsymbol{\psi}} = \ddot{\mathbf{u}} - {}_{(0)}h^-({}_{(0)}\ddot{\mathbf{t}} - {}_{(-1)}\ddot{\mathbf{t}}). \tag{72}$$

Substituting ${}_{(\ell)}\dot{\boldsymbol{\psi}}$ from eqns (67), (69) and (70) into eqn (65), we obtain the alternative form of the rate of the kinetic energy as

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{K} = \sum_{\ell=-1}^{1}\int_{\mathscr{A}} [{}_{(\ell)}\mathbf{f} \cdot {}_{(\ell)}\dot{\boldsymbol{\psi}} + {}_{(\ell)}\mathbf{q} \cdot {}_{(\ell)}\dot{\mathbf{t}}]\, \mathrm{d}\mathscr{A},$$

$$= \int_{\mathscr{A}} \langle \{ {}_{(-1)}\mathbf{f} \cdot [\dot{\mathbf{u}} - {}_{(0)}h^-({}_{(0)}\dot{\mathbf{t}} - {}_{(-1)}\dot{\mathbf{t}})] + {}_{(-1)}\mathbf{q} \cdot {}_{(-1)}\dot{\mathbf{t}} \}$$

$$+ ({}_{(0)}\mathbf{f} \cdot \dot{\mathbf{u}} + {}_{(0)}\mathbf{q} \cdot {}_{(0)}\dot{\mathbf{t}}) + \{ {}_{(1)}\mathbf{f} \cdot [\dot{\mathbf{u}} + {}_{(0)}h^+({}_{(0)}\dot{\mathbf{t}} - {}_{(1)}\dot{\mathbf{t}})\}$$

$$+ {}_{(1)}\mathbf{q} \cdot {}_{(1)}\dot{\mathbf{t}} \} \rangle\, \mathrm{d}\mathscr{A}$$

$$= \int_{\mathscr{A}} \left\{ \left(\sum_{\ell=-1}^{1} {}_{(\ell)}\mathbf{f} \right) \cdot \dot{\mathbf{u}} + ({}_{(-1)}\mathbf{q} + {}_{(0)}h^-\, {}_{(-1)}\mathbf{f}) \cdot {}_{(-1)}\dot{\mathbf{t}} \right.$$

$$+ [{}_{(0)}\mathbf{q} + ({}_{(0)}h^+\, {}_{(1)}\mathbf{f} - {}_{(0)}h^-\, {}_{(-1)}\mathbf{f})] \cdot {}_{(0)}\dot{\mathbf{t}}$$

$$\left. + ({}_{(1)}\mathbf{q} - {}_{(0)}h^+\, {}_{(1)}\mathbf{f}) \cdot {}_{(1)}\dot{\mathbf{t}} \right\} \mathrm{d}\mathscr{A}. \tag{73}$$

From eqn (73), we define the total inertia force $\hat{\mathbf{f}}$ as

$$\hat{\mathbf{f}} := \sum_{\ell=-1}^{1} {}_{(\ell)}\mathbf{f}, \tag{74}$$

in inertia couple ${}_{(\ell)}\mathbf{C}$ for layer (ℓ), with $\ell = 1, 0, -1$, as

$$\begin{aligned}
{}_{(1)}\mathbf{C} &:= {}_{(1)}\mathbf{q} - {}_{(0)}h^{+}\,{}_{(1)}\mathbf{f},\\
{}_{(0)}\mathbf{C} &:= {}_{(0)}\mathbf{q} + ({}_{(0)}h^{+}\,{}_{(1)}\mathbf{f} - {}_{(0)}h^{-}\,{}_{(-1)}\mathbf{f}),\\
{}_{(-1)}\mathbf{C} &:= {}_{(-1)}\mathbf{q} + {}_{(0)}h^{-}\,{}_{(-1)}\mathbf{f}.
\end{aligned} \tag{75}$$

We also define the kth mass moment per unit underformed area of all the three layers as

$$\hat{A}_{\rho}^{k} = \sum_{\ell=-1}^{1} {}_{(\ell)}A_{\rho}^{k}, \quad \text{for } k = 0, 1, 2. \tag{76}$$

From eqns $(66)_1$ and (74), we obtain the total inertia force as

$$\begin{aligned}
\hat{\mathbf{f}} = \sum_{\ell=-1}^{1} {}_{(\ell)}\mathbf{f} &= \sum_{\ell=-1}^{1} ({}_{(\ell)}A_{\rho\,(\ell)}^{0}\ddot{\boldsymbol{\psi}} + {}_{(\ell)}A_{\rho\,(\ell)}^{1}\ddot{\mathbf{t}})\\
&= ({}_{(1)}A_{\rho\,(1)}^{0}\ddot{\boldsymbol{\psi}} + {}_{(1)}A_{\rho\,(1)}^{1}\ddot{\mathbf{t}}) + ({}_{(0)}A_{\rho\,(0)}^{0}\ddot{\boldsymbol{\psi}} + {}_{(0)}A_{\rho\,(0)}^{1}\ddot{\mathbf{t}})\\
&\quad + ({}_{(-1)}A_{\rho\,(-1)}^{0}\ddot{\boldsymbol{\psi}} + {}_{(-1)}A_{\rho\,(-1)}^{1}\ddot{\mathbf{t}})\\
&= {}_{(1)}A_{\rho}^{0}[\ddot{\mathbf{u}} + {}_{(0)}h^{+}({}_{(0)}\ddot{\mathbf{t}} - {}_{(1)}\ddot{\mathbf{t}})] + {}_{(1)}A_{\rho\,(1)}^{1}\ddot{\mathbf{t}} + {}_{(0)}A_{\rho}^{0}\ddot{\mathbf{u}} + {}_{(0)}A_{\rho\,(0)}^{1}\ddot{\mathbf{t}}\\
&\quad + {}_{(-1)}A_{\rho}^{0}[\ddot{\mathbf{u}} - {}_{(0)}h^{-}({}_{(0)}\ddot{\mathbf{t}} - {}_{(-1)}\ddot{\mathbf{t}})] + {}_{(-1)}A_{\rho\,(-1)}^{1}\ddot{\mathbf{t}}\\
&= \hat{A}_{\rho}^{0}\ddot{\mathbf{u}} + [-{}_{(0)}h^{-}\,{}_{(1)}A_{\rho}^{0} + {}_{(1)}A_{\rho}^{1}]_{(1)}\ddot{\mathbf{t}}\\
&\quad + [-{}_{(0)}h^{-}\,{}_{(-1)}A_{\rho}^{0} + {}_{(0)}A_{\rho}^{1} + {}_{(0)}h^{-}\,{}_{(1)}A_{\rho}^{0}]_{(0)}\ddot{\mathbf{t}}\\
&\quad + [{}_{(0)}h^{-}\,{}_{(-1)}A_{\rho}^{0} + {}_{(-1)}A_{\rho}^{1}]_{(-1)}\ddot{\mathbf{t}},
\end{aligned} \tag{78}$$

and the inertia couple ${}_{(\ell)}\mathbf{C}$ for layer (ℓ) as

$$\begin{aligned}
{}_{(1)}\mathbf{C} &= [-{}_{(1)}A_{\rho\,(0)}^{0}h^{+}\,{}_{(1)}\ddot{\boldsymbol{\psi}} + {}_{(1)}A_{\rho}^{1}({}_{(1)}\ddot{\boldsymbol{\psi}} - {}_{(0)}h^{+}\,{}_{(1)}\ddot{\mathbf{t}}) + {}_{(1)}A_{\rho\,(1)}^{2}\ddot{\mathbf{t}}]\\
&= \{-{}_{(1)}A_{\rho\,(0)}^{0}h^{+}[\dot{\mathbf{u}} + {}_{(0)}h^{+}({}_{(0)}\ddot{\mathbf{t}} - {}_{(1)}\ddot{\mathbf{t}})]\\
&\quad + {}_{(1)}A_{\rho}^{1}[\ddot{\mathbf{u}} + {}_{(0)}h^{+}({}_{(0)}\ddot{\mathbf{t}} - {}_{(1)}\ddot{\mathbf{t}}) - {}_{(0)}h^{-}\,{}_{(1)}\ddot{\mathbf{t}}] + {}_{(1)}A_{\rho\,(1)}^{2}\ddot{\mathbf{t}}\}\\
&= [{}_{(0)}h^{+}({}_{(0)}h^{+}\,{}_{(1)}A_{\rho}^{0} - 2{}_{(1)}A_{\rho}^{1}) + {}_{(1)}A_{\rho}^{2}]_{(1)}\ddot{\mathbf{t}}\\
&\quad + (-{}_{(0)}h^{-}\,{}_{(1)}A_{\rho}^{0} + {}_{(1)}A_{\rho}^{1})(\ddot{\mathbf{u}} + {}_{(0)}h^{+}\,{}_{(0)}\ddot{\mathbf{t}}),
\end{aligned} \tag{79}$$

$$\begin{aligned}
{}_{(0)}\mathbf{C} &= [(-{}_{(0)}h^{-}\,{}_{(-1)}A_{\rho\,(-1)}^{0}\ddot{\boldsymbol{\psi}} + {}_{(0)}h^{+}\,{}_{(1)}A_{\rho\,(1)}^{0}\ddot{\boldsymbol{\psi}})\\
&\quad + (-{}_{(0)}h^{-}\,{}_{(-1)}A_{\rho\,(-1)}^{1}\ddot{\mathbf{t}} + {}_{(0)}A_{\rho\,(0)}^{1}\ddot{\boldsymbol{\psi}}) + ({}_{(0)}h^{+}\,{}_{(1)}A_{\rho\,(1)}^{1}\ddot{\mathbf{t}} + {}_{(0)}A_{\rho\,(0)}^{2}\ddot{\mathbf{t}})]\\
&= -{}_{(0)}h^{-}\,{}_{(-1)}A_{\rho}^{0}[\ddot{\mathbf{u}} - {}_{(0)}h^{-}({}_{(0)}\ddot{\mathbf{t}} - {}_{(-1)}\ddot{\mathbf{t}})]\\
&\quad + {}_{(0)}h^{+}\,{}_{(1)}A_{\rho}^{0}[\ddot{\mathbf{u}} + {}_{(0)}h^{+}({}_{(0)}\ddot{\mathbf{t}} - {}_{(1)}\ddot{\mathbf{t}})]\\
&\quad + (-{}_{(0)}h^{-}\,{}_{(-1)}A_{\rho\,(-1)}^{1}\ddot{\mathbf{t}} + {}_{(0)}A_{\rho}^{1}\ddot{\mathbf{u}}) + {}_{(0)}h^{+}\,{}_{(1)}A_{\rho\,(1)}^{1}\ddot{\mathbf{t}} + {}_{(0)}A_{\rho\,(0)}^{2}\ddot{\mathbf{t}}\\
&= {}_{(0)}h^{+}({}_{(1)}A_{\rho}^{1} - {}_{(0)}h^{+}\,{}_{(1)}A_{\rho}^{0})_{(1)}\ddot{\mathbf{t}}
\end{aligned}$$

$$
\begin{aligned}
&+[({}_{(0)}h^{-})^2\,{}_{(-1)}A^0_\rho+({}_{(0)}h^{+})^2\,{}_{(1)}A^0_\rho+{}_{(0)}A^2_\rho]_{(0)}\ddot{\mathbf{t}} \\
&+{}_{(0)}h^{-}\,(-{}_{(-1)}A^1_\rho-{}_{(0)}h^{-}\,{}_{(-1)}A^0_\rho)_{(-1)}\ddot{\mathbf{t}} \\
&+(-{}_{(0)}h^{-}\,{}_{(-1)}A^0_\rho+{}_{(0)}A^1_\rho+{}_{(0)}h^{+}\,{}_{(1)}A^0_\rho)\ddot{\mathbf{u}},
\end{aligned} \tag{80}
$$

$$
\begin{aligned}
{}_{(-1)}\mathbf{C} &= [{}_{(-1)}A^0_{\rho\,(0)}h^{-}\,{}_{(-1)}\ddot{\boldsymbol{\psi}}+{}_{(-1)}A^1_\rho({}_{(-1)}\ddot{\boldsymbol{\psi}}+{}_{(0)}h^{-}\,{}_{(-1)}\ddot{\mathbf{t}})+{}_{(-1)}A^2_{\rho\,(-1)}\ddot{\mathbf{t}}] \\
&= \{{}_{(-1)}A^0_{\rho\,(0)}h^{-}[\ddot{\mathbf{u}}-{}_{(0)}h^{-}({}_{(0)}\ddot{\mathbf{t}}-{}_{(-1)}\ddot{\mathbf{t}})] \\
&\quad +{}_{(-1)}A^1_\rho[\ddot{\mathbf{u}}-{}_{(0)}h^{-}({}_{(0)}\ddot{\mathbf{t}}-{}_{(-1)}\ddot{\mathbf{t}})+{}_{(0)}h^{-}\,{}_{(-1)}\ddot{\mathbf{t}}]+{}_{(-1)}A^2_{\rho\,(-1)}\ddot{\mathbf{t}}\} \\
&= ({}_{(0)}h^{-}\,{}_{(-1)}A^0_\rho+{}_{(-1)}A^1_\rho)(\ddot{\mathbf{u}}-{}_{(0)}h^{-}\,{}_{(0)}\ddot{\mathbf{t}}) \\
&\quad +[{}_{(0)}h^{-}({}_{(0)}h^{-}\,{}_{(-1)}A^0_\rho+2\,{}_{(-1)}A^1_\rho)+{}_{(-1)}A^2_\rho]_{(-1)}\ddot{\mathbf{t}}.
\end{aligned} \tag{81}
$$

The rate of kinetic energy (73) of the sandwich shell can finally be written as

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{K} = \int_{\mathscr{A}}\left[\hat{\mathbf{f}}\cdot\dot{\mathbf{u}}+\sum_{\ell=-1}^{1}{}_{(\ell)}\mathbf{C}\cdot{}_{(\ell)}\dot{\mathbf{t}}\right]\mathrm{d}\mathscr{A}. \tag{82}
$$

3.4. *Balance of power: equations of motion for sandwich shells*

The balance of the total power of the sandwich shell is given by

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{K}+\mathscr{P}_{\mathrm{c}} = \mathscr{P}_{\mathrm{a}}. \tag{83}
$$

In deriving the equation of motion, we will not apply the three constraints of the inextensibility of the directors; therefore there will be 12 components of virtual velocities, and a total of 12 corresponding scalar equations of motion in component form. The three constraints of the inextensibility of the directors will be used in the weak form [see Simo and Fox (1989) for single layer shells, and Vu-Quoc and Deng (1995b) for multilayer shells].

Substituting in eqn (83) the time rate of the kinetic energy eqn (82), the power of the contact forces/couples eqn (55), and the power of the assigned forces/couples eqn (56), and treating $\dot{\mathbf{u}}$ and ${}_{(\ell)}\dot{\mathbf{t}}$ as virtual velocities, we obtain the following equations of motion [see also Vu-Quoc and Ebcioğlu (1995) for multilayer beams and one-dimensional plates].

$$
\begin{aligned}
\hat{\mathbf{n}}^{\alpha}{}_{,\alpha}+\mathbf{n}^* &= \hat{\mathbf{f}}, \\
({}_{(1)}\tilde{\mathbf{m}}^{\alpha}+{}_{(1)}h^{-}\,{}_{(1)}\mathbf{n}^{\alpha})_{,\alpha}-{}_{(1)}\boldsymbol{\ell}+{}_{(1)}\tilde{\mathbf{m}}^* &= {}_{(1)}\mathbf{C}, \\
({}_{(0)}\tilde{\mathbf{m}}^{\alpha}+{}_{(0)}h^{+}\,{}_{(1)}\mathbf{n}^{\alpha}-{}_{(0)}h^{-}\,{}_{(-1)}\mathbf{n}^{\alpha})_{,\alpha}-{}_{(0)}\boldsymbol{\ell}+{}_{(0)}\tilde{\mathbf{m}}^* &= {}_{(0)}\mathbf{C}, \\
({}_{(-1)}\tilde{\mathbf{m}}^{\alpha}-{}_{(-1)}h^{+}\,{}_{(-1)}\mathbf{n}^{\alpha})_{,\alpha}-{}_{(-1)}\boldsymbol{\ell}+{}_{(-1)}\tilde{\mathbf{m}}^* &= {}_{(-1)}\mathbf{C},
\end{aligned} \tag{84}
$$

where eqn $(84)_1$ is the balance of linear momentum referred to the neutral surface of the reference layer (0), and $(84)_2$–$(84)_4$ are the balances of angular momentum of layer (ℓ), for $\ell = -1, 0, 1$, respectively.

The corresponding boundary conditions are obtained from eqn (83), after substitution of eqns (82), (55) and (56), as follows

$$
\begin{aligned}
&\text{either}\quad \mathbf{u} = \mathbf{u}^* \quad \text{or} \quad \hat{\mathbf{n}}^{\alpha} = \mathbf{n}^{*\alpha}, \\
&\text{either}\quad {}_{(1)}\mathbf{t} = {}_{(1)}\mathbf{t}^* \quad \text{or} \quad {}_{(1)}\tilde{\mathbf{m}}^{\alpha}+{}_{(1)}h^{-}\,{}_{(1)}\mathbf{n}^{\alpha} = {}_{(1)}\tilde{\mathbf{m}}^{*\alpha}, \\
&\text{either}\quad {}_{(0)}\mathbf{t} = {}_{(0)}\mathbf{t}^* \quad \text{or} \quad {}_{(0)}\tilde{\mathbf{m}}^{\alpha}-{}_{(0)}h^{-}\,{}_{(-1)}\mathbf{n}^{\alpha}+{}_{(0)}h^{-}\,{}_{(1)}\mathbf{n}^{\alpha} = {}_{(0)}\tilde{\mathbf{m}}^{*\alpha}, \\
&\text{either}\quad {}_{(-1)}\mathbf{t}^* = {}_{(-1)}\mathbf{t}^* \quad \text{or} \quad {}_{(-1)}\tilde{\mathbf{m}}^{\alpha}-{}_{(-1)}h^{+}\,{}_{(-1)}\mathbf{n}^{\alpha} = {}_{(-1)}\tilde{\mathbf{m}}^{*\alpha}.
\end{aligned} \tag{85}
$$

Remark 3.3. Note that eqns (84) and (85) are obtained without the use of the three constraints of inextensibility of the director; there are thus nine scalar equations of balance of angular momentum, i.e. $(84)_2$–$(84)_4$, and the corresponding nine scalar equations in the boundary conditions, i.e. $(85)_2$–$(85)_4$. The three constraints of inextensibility will be introduced in the weak form to reduce the number of equations of balance of angular momentum from nine to six [see Vu-Quoc and Deng (1995b)]. ■

Remark 3.4. From the definition eqn (41) of the director couple ${}_{(\ell)}\ell$, it can be seen that ${}_{(\ell)}\ell$ has the same dimension as the force ${}_{(\ell)}\mathbf{n}^\alpha$ defined in eqn (39) thus $[{}_{(\ell)}\ell] = [{}_{(\ell)}\mathbf{n}^\alpha] = F/L$, i.e. force per unit length, where $[\cdot]$ denotes the dimension of the quantity enclosed inside the square bracket. For this reason, the quantity ${}_{(\ell)}\ell$ is sometimes called the "through-the-thickness resultant force" as in Simo and Fox (1989). However, from the balance of angular momentum equations $(84)_2$–$(84)_4$, ${}_{(\ell)}\ell$ appears together with the assigned couples ${}_{(\ell)}\tilde{\mathbf{m}}^*$, which has the dimension of couple per unit area, i.e. $[{}_{(\ell)}\tilde{\mathbf{m}}^*] = FL/L^2$. Thus it is better to view ${}_{(\ell)}\ell$ as having the dimension of couple per unit area, i.e. $[{}_{(\ell)}\ell] = FL/L^2$, and therein lies the reason for its name, i.e. director couple per unit (deformed) area [see Naghdi (1972, p. 482)]. Of course, the other terms in $(84)_2$–$(84)_4$ also have the same dimension. For example, from the definition (40), the resultant couple ${}_{(\ell)}\tilde{\mathbf{m}}^\alpha$ has the dimension of couple per unit length, i.e. $[{}_{(\ell)}\tilde{\mathbf{m}}^\alpha] = FL/L$, thus the terms ${}_{(\ell)}\tilde{\mathbf{m}}^\alpha_{,\alpha}$ in $(84)_2$–$(84)_4$ have dimension $[{}_{(\ell)}\tilde{\mathbf{m}}^\alpha_{,\alpha}] = FL/L^2$. The terms ${}_{(\ell)}h^{\pm}\,{}_{(\ell)}\mathbf{n}^\alpha_{,\alpha}$ also have dimension FL/L^2, etc. ■

The equations of motion (84) and their corresponding boundary conditions (85) can be simplified if we introduce the following notations

$$
\begin{aligned}
{}_{(1)}\mathcal{M}^\alpha &:= {}_{(1)}\tilde{\mathbf{m}}^\alpha + {}_{(1)}h^-\,{}_{(1)}\mathbf{n}^\alpha,\\
{}_{(0)}\mathcal{M}^\alpha &:= {}_{(0)}\tilde{\mathbf{m}}^\alpha + {}_{(0)}h^+\,{}_{(1)}\mathbf{n}^\alpha - {}_{(0)}h^-\,{}_{(-1)}\mathbf{n}^\alpha,\\
{}_{(-1)}\mathcal{M}^\alpha &:= {}_{(-1)}\tilde{\mathbf{m}}^\alpha - {}_{(-1)}h^+\,{}_{(-1)}\mathbf{n}^\alpha.
\end{aligned}
\tag{86}
$$

Then the equations of motion (84) take the following alternative form:

$$
\begin{aligned}
\hat{\mathbf{n}}^\alpha{}_{,\alpha} + \mathbf{n}^* &= \hat{\mathbf{f}},\\
({}_{(1)}\mathcal{M}^\alpha)_{,\alpha} - {}_{(1)}\ell + {}_{(1)}\tilde{\mathbf{m}}^* &= {}_{(1)}\mathbf{C},\\
({}_{(0)}\mathcal{M}^\alpha)_{,\alpha} - {}_{(0)}\ell + {}_{(0)}\tilde{\mathbf{m}}^* &= {}_{(0)}\mathbf{C},\\
({}_{(-1)}\mathcal{M}^\alpha)_{,\alpha} - {}_{(-1)}\ell + {}_{(-1)}\tilde{\mathbf{m}}^* &= {}_{(-1)}\mathbf{C}.
\end{aligned}
\tag{87}
$$

The force boundary conditions (85) take the following form

$$
\begin{aligned}
\hat{\mathbf{n}}^\alpha &= \mathbf{n}^{*\alpha},\\
{}_{(1)}\mathcal{M}^\alpha &= {}_{(1)}\tilde{\mathbf{m}}^{*\alpha},\\
{}_{(0)}\mathcal{M}^\alpha &= {}_{(0)}\tilde{\mathbf{m}}^{*\alpha},\\
{}_{(-1)}\mathcal{M}^\alpha &= {}_{(-1)}\tilde{\mathbf{m}}^{*\alpha},
\end{aligned}
\tag{88}
$$

Remark 3.5. If (i) only the core layer (0) exists without the outer layers, and (ii) the reference surface we chose is the neutral surface for the sandwich shell [see also (3.22) Simo and Fox (1989)], then the equations of motion for sandwich shells (84) or (87) reduce exactly to those obtained for the single-layer shells in Simo and Fox (1989), keeping in mind the difference in the definition of the resultant forces/couples between the present work and Simo and Fox (1989) [see eqns (39)–(41)].

The above assertion can be proved via the following procedures: (i) all the items that relate to the two outer layers are zero, thus the equation of motion for linear momentum $(84)_1$ becomes

$$
{}_{(0)}\mathbf{n}^{\alpha}{}_{,\alpha} + {}_{(0)}\mathbf{n}^{*} = \hat{\mathbf{f}}, \tag{89}
$$

where the resultant force $\hat{\mathbf{f}}$ is obtained from eqn (78)

$$
\hat{\mathbf{f}} = \hat{A}^{0}_{\rho}\ddot{\mathbf{u}} + {}_{(0)}A^{1}_{\rho\,(0)}\ddot{\mathbf{t}}, \tag{90}
$$

(ii) omit the layer index and remember the definition of the neutral surface leads to ${}_{(0)}A^{1}_{\rho} = 0$, eqn (89), can be reduced to

$$
\mathbf{n}^{\alpha}{}_{,\alpha} + \mathbf{n}^{*} = A^{0}_{\rho}\ddot{\mathbf{u}}. \tag{91}
$$

Notice the difference from our definition of the resultant contact force and that in Simo and Fox (1989), we could see that the equation of motion for linear momentum $(84)_1$ is exactly the same as that given in Simo and Fox (1989).

If we omit the layer index, then the angular momentum equation for the core layer becomes

$$
\tilde{\mathbf{m}}^{\alpha}{}_{,\alpha} - \ell + \tilde{\mathbf{m}}^{*} = \mathbf{C}. \tag{92}
$$

From eqn (80), and use the two assumptions at the beginning of this Remark, we obtain $\mathbf{C} = A^{2}_{\rho}\ddot{\mathbf{t}}$. Recalling the difference from our definition of the resultant contact couples and that in Simo and Fox (1989) (see Remark (3.1)), one sees that the equation of motion for angular momentum $(84)_3$ is exactly the same as that given in Simo and Fox (1989). ■

Remark 3.6. If we assume, on the other hand, that (i) the three layers have the same rotation, i.e.

$$
{}_{(-1)}\mathbf{t} = {}_{(0)}\mathbf{t} = {}_{(1)}\mathbf{t} = \mathbf{t}; \tag{93}
$$

(ii) the reference surface chosen is the neutral surface of the whole sandwich shell i.e.

$$
\sum_{\ell=-1}^{1} \int_{(\ell)\mathscr{H}} {}_{(\ell)}\rho_{(\ell)}j_t\, \xi^3 \,\mathrm{d}\xi^3 = 0; \tag{94}
$$

then the equations of motion (84) for sandwich shells reduce exactly to the equations of motion (128) for single-layer shell in Simo and Fox (1989).

We will first prove that the balance of linear momentum $(84)_1$ for sandwich shells will reduce to that of single-layer shells. From eqn (39), the total contact force for the sandwich shell, eqn (44) can be written as

$$
\hat{\mathbf{n}}^{\alpha} := \sum_{\ell=-1}^{1} {}_{(\ell)}\mathbf{n}^{\alpha} = \sum_{\ell=-1}^{1} \int_{(\ell)\mathscr{H}} {}_{(\ell)}j_{(\ell)}\mathbf{g}^{\alpha}\cdot{}_{(\ell)}\boldsymbol{\sigma}\,\mathrm{d}\xi^3 = \int_{\mathscr{H}} j_t \mathbf{g}^{\alpha}\cdot\boldsymbol{\sigma}\,\mathrm{d}\xi^3. \tag{95}
$$

Using eqns (64), (76) and (94), the inertia force in eqn (78) can be written as

$$
\begin{aligned}
\hat{\mathbf{f}} &= \hat{A}^{0}_{\rho}\ddot{\mathbf{u}} + [{}_{(1)}A^{1}_{\rho} + {}_{(0)}A^{1}_{\rho} + {}_{(-1)}A^{1}_{\rho}]\ddot{\mathbf{t}} \\
&= \left[\sum_{\ell=-1}^{1}\int_{(\ell)\mathscr{H}} {}_{(\ell)}\rho_{(\ell)}j_t\,\mathrm{d}\xi^3\right]\ddot{\mathbf{u}} + \left[\sum_{\ell=-1}^{1}\int_{(\ell)\mathscr{H}} {}_{(\ell)}\rho_{(\ell)}j_t(\xi^3)^1\,\mathrm{d}\xi^3\right]\ddot{\mathbf{t}} \\
&= \left(\int_{\mathscr{H}} \rho j_t\,\mathrm{d}\xi^3\right)\ddot{\mathbf{u}}.
\end{aligned} \tag{96}
$$

Thus, eqn $(84)_1$ reduces to

$$\frac{1}{\bar{j}_t}\left(\int_{\mathscr{H}} j_t \mathbf{g}^{\alpha}\cdot\boldsymbol{\sigma}\,\mathrm{d}\xi^3\right)_{,\alpha} + \frac{1}{\bar{j}_t}\mathbf{n}^* = \frac{1}{\bar{j}_t}\left(\int_{\mathscr{H}} \rho j_t\,\mathrm{d}\xi^3\right)\ddot{\mathbf{u}}. \tag{97}$$

Note that our definition of the applied resultant force $\mathbf{n}^*$ is different, by a factor $1/\bar{j}_t$, from the definition of the applied resultant force per unit length $\bar{\mathbf{n}}$ in equation (A.5) of Simo and Fox (1989). Using eqns (3.23) and (4.5a) of Simo and Fox (1989) for a definition of the mass per unit length $\bar{\rho}$ and the resultant force $\mathbf{n}^{\alpha}$, respectively, we can see that eqn (97) is exactly the same as eqn (4.19a) of Simo and Fox (1989).

We now check the balance of angular momentum. Sum up $(84)_2$–$(84)_4$ to obtain

$$\left[\sum_{\ell=-1}^{1} {}_{(\ell)}\tilde{\mathbf{m}}^{\alpha} + {}_{(1)}h^{-}\,{}_{(1)}\mathbf{n}^{\alpha} + {}_{(0)}h^{+}\,{}_{(1)}\mathbf{n}^{\alpha} - {}_{(0)}h^{-}\,{}_{(-1)}\mathbf{n}^{\alpha} - {}_{(-1)}h^{+}\,{}_{(-1)}\mathbf{n}^{\alpha}\right]_{,\alpha} - \sum_{\ell=-1}^{1} {}_{(\ell)}\ell + \sum_{\ell=-1}^{1} {}_{(\ell)}\tilde{\mathbf{m}}^* = \sum_{\ell=-1}^{1} {}_{(\ell)}\mathbf{C}. \tag{98}$$

Note that

$${}_{(0)}h^{-} + {}_{(1)}h^{-} = {}_{(1)}Z, \quad {}_{(0)}h^{-} + {}_{(-1)}h^{+} = {}_{(-1)}Z, \tag{99}$$

and using the definition, eqns (39) and (40) of resultant forces and couples, the items in the square bracket in eqn (98) can be reduced as

$$\begin{aligned}
&\sum_{\ell=-1}^{1} {}_{(\ell)}\tilde{\mathbf{m}}^{\alpha} + {}_{(1)}h^{-}\,{}_{(1)}\mathbf{n}^{\alpha} + {}_{(0)}h^{+}\,{}_{(1)}\mathbf{n}^{\alpha} - {}_{(0)}h^{-}\,{}_{(-1)}\mathbf{n}^{\alpha} - {}_{(-1)}h^{+}\,{}_{(-1)}\mathbf{n}^{\alpha} \\
&\quad = \sum_{\ell=-1}^{1} {}_{(\ell)}\tilde{\mathbf{m}}^{\alpha} + {}_{(1)}Z\,{}_{(1)}\mathbf{n}^{\alpha} - {}_{(-1)}Z\,{}_{(-1)}\mathbf{n}^{\alpha} = \sum_{\ell=-1}^{1}\int_{(\ell)\mathscr{H}} {}_{(\ell)}j_{t(\ell)}\mathbf{g}^{\alpha}\xi^3\cdot{}_{(\ell)}\boldsymbol{\sigma}\,\mathrm{d}\xi^3 = \int_{\mathscr{H}} j_t\mathbf{g}^{\alpha}\xi^3\cdot\boldsymbol{\sigma}\,\mathrm{d}\xi^3.
\end{aligned} \tag{100}$$

Using eqn (41), the second part in eqn (98) can be written as

$$\sum_{\ell=-1}^{1} {}_{(\ell)}\ell = \sum_{\ell=-1}^{1}\int_{(\ell)\mathscr{H}} {}_{(\ell)}j_{t(\ell)}\mathbf{g}^{3}\cdot{}_{(\ell)}\boldsymbol{\sigma}\,\mathrm{d}\xi^3 = \int_{\mathscr{H}} j_t\mathbf{g}^{3}\cdot\boldsymbol{\sigma}\,\mathrm{d}\xi^3. \tag{101}$$

With equations (79)–(81), (93) and (94), the total inertia couple can be rewritten as

$$\begin{aligned}
\sum_{\ell=-1}^{1} {}_{(\ell)}\mathbf{C} &= [{}_{(0)}h^{+}({}_{(0)}h^{+}\,{}_{(1)}A^0_{\rho} - 2\,{}_{(1)}A^1_{\rho}) + {}_{(0)}A^2_{\rho}]\,{}_{(1)}\ddot{\mathbf{t}} \\
&\quad + (-{}_{(0)}h^{+}\,{}_{(1)}A^0_{\rho} + {}_{(1)}A^1_{\rho})(\ddot{\mathbf{u}} + {}_{(0)}h^{+}\,{}_{(0)}\ddot{\mathbf{t}}) + {}_{(0)}h^{+}({}_{(1)}A^1_{\rho} - {}_{(0)}h^{-}\,{}_{(1)}A^0_{\rho})\,{}_{(1)}\ddot{\mathbf{t}} \\
&\quad + [({}_{(0)}h^{-})^2\,{}_{(-1)}A^0_{\rho} + ({}_{(0)}h^{+})^2\,{}_{(1)}A^0_{\rho} + {}_{(0)}A^2_{\rho}]\,{}_{(0)}\ddot{\mathbf{t}} \\
&\quad + {}_{(0)}h^{-}(-{}_{(-1)}A^1_{\rho} - {}_{(0)}h^{-}\,{}_{(-1)}A^0_{\rho})\,{}_{(-1)}\ddot{\mathbf{t}} + (-{}_{(0)}h^{-}\,{}_{(-1)}A^0_{\rho} + {}_{(0)}A^1_{\rho} + {}_{(0)}h^{+}\,{}_{(1)}A^0_{\rho})\ddot{\mathbf{u}} \\
&\quad + ({}_{(0)}h^{-}\,{}_{(-1)}A^0_{\rho} + {}_{(-1)}A^1_{\rho})(\ddot{\mathbf{u}} - {}_{(0)}h^{-}\,{}_{(0)}\ddot{\mathbf{t}}) \\
&\quad + [{}_{(0)}h^{-}({}_{(0)}h^{-}\,{}_{(-1)}A^0_{\rho} + 2\,{}_{(-1)}A^1_{\rho}) + {}_{(-1)}A^2_{\rho}]\,{}_{(-1)}\ddot{\mathbf{t}} \\
&= \left(\sum_{\ell=-1}^{1} {}_{(\ell)}A^1_{\rho}\right)\ddot{\mathbf{u}} + \left(\sum_{\ell=-1}^{1} {}_{(\ell)}A^2_{\rho}\right)\ddot{\mathbf{t}} = \left(\sum_{\ell=-1}^{1} {}_{(\ell)}A^2_{\rho}\right)\ddot{\mathbf{t}}.
\end{aligned} \tag{102}$$

Finally, using eqns (100)–(102), we can write the balance of total angular momentum (98) as

$$\left(\int_{\mathscr{H}} j_l \mathbf{g}^{\alpha} \xi^3 \cdot \boldsymbol{\sigma}\, \mathrm{d}\xi^3\right)_{,\alpha} - \int_{\mathscr{H}} j_l \mathbf{g}^3 \cdot \boldsymbol{\sigma}\, \mathrm{d}\xi^3 + \sum_{\ell=-1}^{1} {}_{(\ell)}\tilde{\mathbf{m}}^* = \left(\sum_{\ell=-1}^{1} {}_{(\ell)}A_\rho^2\right)\ddot{\mathbf{t}}. \tag{103}$$

Note that our definition of the applied resultant couples $\sum_{\ell=-1}^{1} {}_{(\ell)}\tilde{\mathbf{m}}^*$ is different from the definition of applied director couple per unit length $\tilde{\bar{\mathbf{m}}}$ in eqn (4.18) of Simo and Fox (1989) by a factor $1/\bar{j}_l$. By employing eqns (3.24), (4.10) and (4.11) of Simo and Fox (1989) for definitions of the mass moment of inertia per unit length $\bar{\mathbf{I}}_\rho$, the across-the-thickness resultant ℓ and the resultant couple $\tilde{\mathbf{m}}^\alpha$, we can see that eqn (103) is exactly the same as eqn (4.19b) of Simo and Fox (1989). Thus the equations for single-layer shells constitute a particular case of those for sandwich shells. ■

3.5. *Angular velocity/acceleration of director field*

To solve the equations of motion given in eqn (84), we need the explicit expression for the first and second time rate of the director field. To this end, we recall some mathematical concepts (Simo and Fox, 1989).

Define the set of rotations whose rotation axis is perpendicular to the vector $\mathbf{E}$ as

$$S_{\mathrm{E}}^2 := \{\mathbf{\Lambda} \in SO(3) \,|\, \text{for } \mathbf{\Psi} \in \mathscr{R}^3 \quad \text{such that } \mathbf{\Lambda\Psi} = \mathbf{\Psi}, \quad \text{and} \quad \mathbf{\Psi} \cdot \mathbf{E} = 0\}, \tag{104}$$

where $SO(3)$ is the set of orthogonal transformations. Each rotation operator $\mathbf{\Lambda} \in S_{\mathrm{E}}^2$ will apply a drill-free rotation on the vector $\mathbf{E}$, i.e. a rotation of $\mathbf{E}$ such that $\mathbf{E}$ does not rotate about itself; there is thus no drill d.o.f.s in the rotations. Let $\mathbf{E} = \mathbf{E}^3 = \mathbf{E}_3$ be the material director field. To each spatial director $\mathbf{t} \neq \mathbf{E}$, there exists a unique (Simo and Fox; 1989) rotation operator $\mathbf{\Lambda} \in S_{\mathrm{E}}^2$ such that

$$_{(\ell)}\mathbf{t} = {}_{(\ell)}\mathbf{\Lambda E}. \tag{105}$$

The time rate of the spatial director $\mathbf{t}$ can be expressed as

$$_{(\ell)}\dot{\mathbf{t}} = {}_{(\ell)}\dot{\mathbf{\Lambda}}\mathbf{E} = {}_{(\ell)}\mathbf{\Lambda}({}_{(\ell)}\mathbf{\Lambda}'\,{}_{(\ell)}\dot{\mathbf{\Lambda}})\mathbf{E} = {}_{(\ell)}\dot{\mathbf{\Lambda}}\,{}_{(\ell)}\mathbf{\Lambda}'\,{}_{(\ell)}\mathbf{t}. \tag{106}$$

Define

$$_{(\ell)}\hat{\mathbf{\Omega}} := {}_{(\ell)}\mathbf{\Lambda}'\,{}_{(\ell)}\dot{\mathbf{\Lambda}}, \quad {}_{(\ell)}\hat{\boldsymbol{\omega}} := {}_{(\ell)}\dot{\mathbf{\Lambda}}\,{}_{(\ell)}\mathbf{\Lambda}', \tag{107}$$

where $_{(\ell)}\hat{\mathbf{\Omega}}$ is the material rotation velocity tensor, and $_{(\ell)}\hat{\boldsymbol{\omega}}$ the spatial rotation velocity tensor. Both $_{(\ell)}\hat{\mathbf{\Omega}}$ and $_{(\ell)}\hat{\boldsymbol{\omega}}$ are skew-symmetric [see Simo and Fox (1989)]. We also define the axial vector $_{(\ell)}\boldsymbol{\omega}$ associated with $_{(\ell)}\hat{\boldsymbol{\omega}}$, [see, e.g. Chadwick (1976, p. 29)], as follows

$$\forall \mathbf{h} \in \mathscr{R}^3, \quad {}_{(\ell)}\hat{\boldsymbol{\omega}}\mathbf{h} = {}_{(\ell)}\boldsymbol{\omega} \times \mathbf{h}. \tag{108}$$

Thus the time rate of the director field, eqn (106), can be expressed as

$$_{(\ell)}\dot{\mathbf{t}} = {}_{(\ell)}\boldsymbol{\omega} \times {}_{(\ell)}\mathbf{t}, \quad {}_{(\ell)}\boldsymbol{\omega} \cdot {}_{(\ell)}\mathbf{t} = 0, \tag{109}$$

which is the requirement of no-drill d.o.f.s.

4. LINEARIZED EQUATIONS OF MOTION FOR SANDWICH SHELLS/PLATES

The fully nonlinear stress resultant geometrically-exact sandwich shell equations of motion obtained in Section 3 will be linearized systematically here, based on the small deformation assumption. The reader is referred to Vu-Quoc and Ebcioğlu (1995) for the analogy in the linearization of geometrically-exact sandwich beams and one-dimensional plates. Here we will verify the proposed theory by comparing a particular case of its linearized version to the linear equations of sandwich plates obtained in Yu (1959). In this particular case, a symmetric sandwich plate is considered, in which the top layer (1) is identical to the bottom layer (-1), and the core layer (0) is symmetric about its neutral surface. Further, it is assumed that the rotations in layer (1) and in layer (-1) are the same [see Yu (1959)].

4.1. *Alternative equations of motion for sandwich shell*

For the subsequent comparison of the linearized equations with those in Yu (1959), we can replace the three equations of motion for angular momentum $(84)_2$–$(84)_4$ by a set of equivalent equations. First, summing up eqn $(84)_2$–$(84)_4$, we obtain the equations of motion for total angular momentum of geometrically-exact sandwich shells

$$[\hat{\tilde{\mathbf{m}}}^{\alpha}+({}_{(0)}h^{-}\,{}_{(1)}\mathbf{n}^{\alpha}-{}_{(0)}h^{-}\,{}_{(-1)}\mathbf{n}^{\alpha})+({}_{(1)}h^{-}\,{}_{(1)}\mathbf{n}^{\alpha}-{}_{(-1)}h^{+}\,{}_{(-1)}\mathbf{n}^{\alpha})]_{,\alpha}-\hat{\ell}+\hat{\tilde{\mathbf{m}}}^{*}=\hat{\mathbf{C}}, \tag{110}$$

where

$$\hat{\tilde{\mathbf{m}}}^{\alpha}:={}_{(1)}\tilde{\mathbf{m}}^{\alpha}+{}_{(0)}\tilde{\mathbf{m}}^{\alpha}+{}_{(-1)}\tilde{\mathbf{m}}^{\alpha}, \tag{111}$$

$$\hat{\ell}:={}_{(1)}\ell^{\alpha}+{}_{(0)}\ell+{}_{(-1)}\ell, \tag{112}$$

$$\hat{\tilde{\mathbf{m}}}^{*}:={}_{(1)}\tilde{\mathbf{m}}^{*}+{}_{(0)}\tilde{\mathbf{m}}^{*}+{}_{(-1)}\tilde{\mathbf{m}}^{*}, \tag{113}$$

$$\hat{\mathbf{C}}:={}_{(1)}\mathbf{C}^{\alpha}+{}_{(0)}\mathbf{C}+{}_{(-1)}\mathbf{C}, \tag{114}$$

with $\hat{\tilde{\mathbf{m}}}^{\alpha}$ being the total contact couple on the facet with normal ${}_{(\ell)}\mathbf{g}^{\alpha}$, $\hat{\ell}$ the total director couple, $\hat{\tilde{\mathbf{m}}}^{*}$ the total assigned couple, and $\hat{\mathbf{C}}$ the total inertia couple. The difference between the contact couple ${}_{(1)}\tilde{\mathbf{m}}^{\alpha}-{}_{(-1)}\tilde{\mathbf{m}}^{\alpha}$ is governed by the following equation

$$\begin{aligned}&[({}_{(1)}\tilde{\mathbf{m}}^{\alpha}-{}_{(-1)}\tilde{\mathbf{m}}^{\alpha})+({}_{(1)}h^{-}\,{}_{(1)}\mathbf{n}^{\alpha}+{}_{(-1)}h^{+}\,{}_{(-1)}\mathbf{n}^{\alpha})]_{,\alpha}\\&\qquad-({}_{(1)}\ell-{}_{(-1)}\ell)+({}_{(1)}\tilde{\mathbf{m}}^{*}-{}_{(-1)}\tilde{\mathbf{m}}^{*})={}_{(1)}\mathbf{C}-{}_{(-1)}\mathbf{C},\end{aligned} \tag{115}$$

obtained by subtracting eqn $(84)_2$ from eqn $(84)_4$. Thus a set of three equations of balance of angular momentum equivalent to eqn $(84)_2$–$(84)_4$ is eqns (110), $(84)_3$, and (115).

The boundary conditions corresponding to (110) and (115) are obtained by adding eqn $(85)_2$–$(85)_4$ and by subtracting eqn $(85)_2$ from eqns $(85)_4$ to yield

$$\hat{\tilde{\mathbf{m}}}^{\alpha}-{}_{(-1)}\mathbf{n}^{\alpha}({}_{(-1)}h^{+}+{}_{(0)}h^{-})+{}_{(1)}\mathbf{n}^{\alpha}({}_{(0)}h^{+}+{}_{(1)}h^{-})=\hat{\tilde{\mathbf{m}}}^{*\alpha}, \tag{116}$$

$$({}_{(1)}\tilde{\mathbf{m}}^{\alpha}-{}_{(-1)}\tilde{\mathbf{m}}^{\alpha})+({}_{(1)}h^{-}\,{}_{(1)}\mathbf{n}^{\alpha}+{}_{(-1)}h^{-}\,{}_{(-1)}\mathbf{n}^{\alpha})={}_{(1)}\tilde{\mathbf{m}}^{*}-{}_{(-1)}\tilde{\mathbf{m}}^{*}. \tag{117}$$

Thus the boundary conditions for the equations of motion (110), $(84)_3$, and (115), are eqns (116), $(85)_3$, and (117), respectively.

From eqns (79)–(81), we obtain the total inertia couple $\hat{\mathbf{C}}$ as follows

$$\begin{aligned}\hat{\mathbf{C}}=\;&\hat{A}^{1}_{\rho}\ddot{\mathbf{u}}+\sum_{\ell=-1}^{1}{}_{(\ell)}A^{2}_{\rho\,(\ell)}\ddot{\mathbf{t}}+{}_{(0)}h^{+}\,{}_{(1)}A^{1}_{\rho}({}_{(0)}\ddot{\mathbf{t}}-2\,{}_{(1)}\ddot{\mathbf{t}})-{}_{(0)}h^{-}\,{}_{(-1)}A^{1}_{\rho}({}_{(0)}\ddot{\mathbf{t}}-2\,{}_{(-1)}\ddot{\mathbf{t}})\\&-({}_{(0)}h^{+})^{2}\,{}_{(1)}A^{0}_{\rho}({}_{(0)}\ddot{\mathbf{t}}-{}_{(1)}\ddot{\mathbf{t}})-({}_{(0)}h^{-})^{2}\,{}_{(-1)}A^{0}_{\rho}({}_{(0)}\ddot{\mathbf{t}}-{}_{(-1)}\ddot{\mathbf{t}})\\&+({}_{(0)}h^{+})^{2}\,{}_{(1)}A^{0}_{\rho}({}_{(0)}\ddot{\mathbf{t}}-{}_{(1)}\ddot{\mathbf{t}})+({}_{(0)}h^{-})^{2}\,{}_{(-1)}A^{0}_{\rho}({}_{(0)}\ddot{\mathbf{t}}-{}_{(-1)}\ddot{\mathbf{t}})\\&+{}_{(0)}h^{+}\,{}_{(1)}A^{1}_{\rho\,(1)}\ddot{\mathbf{t}}-{}_{(0)}h^{-}\,{}_{(-1)}A^{1}_{\rho\,(-1)}\ddot{\mathbf{t}}\\=\;&\hat{A}^{1}_{\rho}\ddot{\mathbf{u}}+\sum_{\ell=-1}^{1}{}_{(\ell)}A^{2}_{\rho\,(\ell)}\ddot{\mathbf{t}}\\&+{}_{(0)}h^{+}\,{}_{(1)}A^{1}_{\rho}[{}_{(0)}\ddot{\mathbf{t}}-{}_{(1)}\ddot{\mathbf{t}}]-{}_{(0)}h^{-}\,{}_{(-1)}A^{1}_{\rho}[{}_{(0)}\ddot{\mathbf{t}}-{}_{(-1)}\ddot{\mathbf{t}}],\end{aligned} \tag{118}$$

and the difference $({}_{(1)}\mathbf{C}-{}_{(-1)}\mathbf{C})$ as follows

$$
\begin{aligned}
{}_{(1)}\mathbf{C}-{}_{(-1)}\mathbf{C} &= (-{}_{(0)}h^{+}\,{}_{(1)}A^0_\rho - {}_{(0)}h^{-}\,{}_{(-1)}A^0_\rho + {}_{(1)}A^1_\rho - {}_{(-1)}A^1_\rho)\ddot{\mathbf{u}} \\
&\quad + {}_{(1)}A^2_{\rho\,(1)}\ddot{\mathbf{t}} - {}_{(-1)}A^2_{\rho\,(-1)}\ddot{\mathbf{t}} + {}_{(0)}h^{+}\,{}_{(1)}A^1_\rho({}_{(0)}\ddot{\mathbf{t}}-2{}_{(1)}\ddot{\mathbf{t}}) + {}_{(0)}h^{-}\,{}_{(-1)}A^1_\rho({}_{(0)}\ddot{\mathbf{t}}-2{}_{(-1)}\ddot{\mathbf{t}}) \\
&\quad - ({}_{(0)}h^{+})^2\,{}_{(1)}A^0_\rho({}_{(0)}\ddot{\mathbf{t}}-{}_{(1)}\ddot{\mathbf{t}}) + ({}_{(0)}h^{-})^2\,{}_{(-1)}A^0_\rho({}_{(0)}\ddot{\mathbf{t}}-{}_{(-1)}\ddot{\mathbf{t}}). \qquad (119)
\end{aligned}
$$

4.2. *Equations of motion for symmetric sandwich shell*

Now, let us consider the symmetric sandwich shell. We have

$$
{}_{(\ell)}h^{+} = {}_{(\ell)}h^{-} = {}_{(\ell)}h, \quad \forall \ell = -1,0,1, \quad {}_{(1)}h = {}_{(-1)}h, \quad {}_{(1)}\rho = {}_{(-1)}\rho, \qquad (120)
$$

and thus

$$
{}_{(0)}A^1_\rho = 0, \quad {}_{(1)}A^0_\rho = {}_{(-1)}A^0_\rho, \quad {}_{(1)}A^1_\rho = -{}_{(-1)}A^1_\rho, \quad {}_{(1)}A^2_\rho = {}_{(-1)}A^2_\rho. \qquad (121)
$$

From eqns (84) and (78) and (79), we obtain the following nonlinear equations of motion for symmetric sandwich shells

$$
\hat{\mathbf{n}}^{\alpha}{}_{,\alpha} + \mathbf{n}^* = ({}_{(0)}h_{(-1)}A^0_\rho + {}_{(-1)}A^1_\rho)_{(-1)}\ddot{\mathbf{t}} + (-{}_{(0)}h_{(1)}A^0_\rho + {}_{(1)}A^1_\rho)_{(1)}\ddot{\mathbf{t}} + \hat{A}^0_\rho\ddot{\mathbf{u}} \qquad (122)
$$

$$
\begin{aligned}
({}_{(1)}\tilde{\mathbf{m}}^{\alpha} + {}_{(1)}h_{(1)}\hat{\mathbf{n}}^{\alpha})_{,\alpha} - {}_{(1)}\ell + {}_{(1)}\tilde{\mathbf{m}}^* &= (-{}_{(0)}h_{(1)}A^0_\rho + {}_{(1)}A^1_\rho)\ddot{\mathbf{u}} + {}_{(1)}A^2_{\rho\,(1)}\ddot{\mathbf{t}} \\
&\quad + {}_{(0)}h_{(1)}A^1_\rho({}_{(0)}\ddot{\mathbf{t}} - 2{}_{(1)}\ddot{\mathbf{t}}) - ({}_{(0)}h)^2\,{}_{(1)}A^0_\rho({}_{(0)}\ddot{\mathbf{t}} - {}_{(1)}\ddot{\mathbf{t}}), \qquad (123)
\end{aligned}
$$

$$
\begin{aligned}
({}_{(0)}\tilde{\mathbf{m}}^{\alpha} + {}_{(0)}h_{(1)}\mathbf{n}^{\alpha} - {}_{(0)}h_{(-1)}\mathbf{n}^{\alpha})_{,\alpha} - {}_{(0)}\ell + {}_{(0)}\tilde{\mathbf{m}}^* &= {}_{(0)}h_{(1)}A^1_{\rho\,(1)}\ddot{\mathbf{t}} - {}_{(0)}h_{(-1)}A^1_{\rho\,(-1)}\ddot{\mathbf{t}} + {}_{(0)}A^2_{\rho\,(0)}\ddot{\mathbf{t}} \\
&\quad + ({}_{(0)}h)^2\,{}_{(1)}A^0_\rho({}_{(0)}\ddot{\mathbf{t}} - {}_{(1)}\ddot{\mathbf{t}}) + ({}_{(0)}h)^2\,{}_{(-1)}A^0_\rho({}_{(0)}\ddot{\mathbf{t}} - {}_{(-1)}\ddot{\mathbf{t}}), \qquad (124)
\end{aligned}
$$

$$
\begin{aligned}
({}_{(-1)}\tilde{\mathbf{m}}^{\alpha} - {}_{(-1)}h_{(-1)}\mathbf{n}^{\alpha})_{,\alpha} - {}_{(-1)}\ell + {}_{(-1)}\tilde{\mathbf{m}}^* &= ({}_{(0)}h_{(-1)}A^0_\rho + {}_{(-1)}A^1_\rho)\ddot{\mathbf{u}} + {}_{(-1)}A^2_{\rho\,(-1)}\ddot{\mathbf{t}} \\
&\quad - {}_{(0)}h_{(-1)}A^1_\rho({}_{(0)}\ddot{\mathbf{t}} - 2{}_{(-1)}\ddot{\mathbf{t}}) - ({}_{(0)}h)^2\,{}_{(-1)}A^0_\rho({}_{(0)}\ddot{\mathbf{t}} - {}_{(-1)}\ddot{\mathbf{t}}). \qquad (125)
\end{aligned}
$$

Similarly to the discussion at the beginning of this section, the balance of angular momentum equations (123)–(125) can be replaced by a set of three equivalent equations: eqn (125) is replaced by the sum of eqns (125), (124) and (123); eqn (124) remains the same; eqn (123) is replaced by subtracting eqn (123) from eqn (125). Summing up eqns (123)–(125) we obtain

$$
\begin{aligned}
[\hat{\tilde{\mathbf{m}}}^{\alpha} + ({}_{(0)}h + {}_{(1)}h)({}_{(1)}\mathbf{n}^{\alpha} - {}_{(-1)}\mathbf{n}^{\alpha})]_{,\alpha} - \hat{\ell} + \hat{\tilde{\mathbf{m}}}^* &= \hat{A}^1_\rho\ddot{\mathbf{u}} + \sum_{\ell=-1}^{1} {}_{(\ell)}A^2_{\rho\,(\ell)}\ddot{\mathbf{t}} + {}_{(0)}h_{(1)}A^1_\rho(2{}_{(0)}\ddot{\mathbf{t}} - {}_{(1)}\ddot{\mathbf{t}} - {}_{(-1)}\ddot{\mathbf{t}}) \\
&= \sum_{\ell=-1}^{1} {}_{(\ell)}A^2_{\rho\,(\ell)}\ddot{\mathbf{t}} + {}_{(0)}h_{(1)}A^1_\rho(2{}_{(0)}\ddot{\mathbf{t}} - {}_{(1)}\ddot{\mathbf{t}} - {}_{(-1)}\ddot{\mathbf{t}}). \qquad (126)
\end{aligned}
$$

Subtracting eqn (123) from eqn (125), we obtain

$$
\begin{aligned}
&[({}_{(1)}\tilde{\mathbf{m}}^{\alpha} - {}_{(-1)}\tilde{\mathbf{m}}^{\alpha}) + {}_{(1)}h({}_{(1)}\mathbf{n}^{\alpha} + {}_{(-1)}\mathbf{n}^{\alpha})]_{,\alpha} - ({}_{(1)}\ell - {}_{(-1)}\ell) + ({}_{(1)}\tilde{\mathbf{m}}^* - {}_{(-1)}\tilde{\mathbf{m}}^*) \\
&\quad = 2(-{}_{(0)}h_{(1)}A^0_\rho + {}_{(1)}A^1_\rho)\ddot{\mathbf{u}} + [{}_{(1)}A^2_\rho - 2{}_{(0)}h_{(1)}A^1_\rho + ({}_{(0)}h)^2\,{}_{(1)}A^0_\rho]({}_{(1)}\ddot{\mathbf{t}} - {}_{(-1)}\ddot{\mathbf{t}}). \qquad (127)
\end{aligned}
$$

The force boundary conditions for eqns (126) and (127) are obtained from eqn (116) and (117), by using the symmetric condition (120), as follows

$$\hat{\tilde{\mathbf{m}}}^{\alpha}+({}_{(0)}h+{}_{(1)}h)({}_{(1)}\mathbf{n}^{\alpha}-{}_{(-1)}\mathbf{n}^{\alpha}) = \hat{\tilde{\mathbf{m}}}^{*}, \tag{128}$$

$$({}_{(1)}\tilde{\mathbf{m}}^{\alpha}-{}_{(-1)}\tilde{\mathbf{m}}^{\alpha})+{}_{(1)}h({}_{(1)}\mathbf{n}^{\alpha}+{}_{(-1)}\mathbf{n}^{\alpha}) = {}_{(1)}\tilde{\mathbf{m}}^{*}-{}_{(-1)}\tilde{\mathbf{m}}^{*}. \tag{129}$$

4.3. *Classical results for symmetric sandwich plate*

The equations of motion obtained in the previous section reduce exactly to the classical results for sandwich plate obtained in Yu (1959), provided we make the following assumptions as in Yu (1959): (i) only small deformation is considered (this assumption only affects the inertia part of the equations of motion); (ii) the sandwich plate is symmetric with respect to the neutral surface, i.e. eqns (120) and (121) hold true; (iii) the two outer layers have the same rotation, i.e. ${}_{(-1)}\mathbf{t} = {}_{(1)}\mathbf{t}$.

A consequence of the assumption (i) above is that ${}_{(\ell)}j_t \approx 1$. From eqns (39) and (44) we obtain the total resultant stress $\hat{\mathbf{n}}^{\alpha}$ as

$$\hat{\mathbf{n}}^{\alpha} := \sum_{\ell=-1}^{1} \int_{{}_{(\ell)}\mathscr{H}} {}_{(\ell)}\mathbf{g}^{\alpha}\cdot{}_{(\ell)}\boldsymbol{\sigma}\,\mathrm{d}\xi^{3}. \tag{130}$$

The resultant couple ${}_{(\ell)}\tilde{\mathbf{m}}^{\alpha}$ and the director couple ${}_{(\ell)}\ell$ can be simplified by substituting ${}_{(\ell)}j_t \approx 1$ into eqns (40) and (41) to yield

$$\begin{aligned} {}_{(1)}\tilde{\mathbf{m}}^{\alpha} &:= \int_{{}_{(1)}\mathscr{H}} (\xi^{3}-{}_{(1)}Z){}_{(1)}\mathbf{g}^{\alpha}\cdot{}_{(1)}\boldsymbol{\sigma}\,\mathrm{d}\xi^{3},\\ {}_{(0)}\tilde{\mathbf{m}}^{\alpha} &:= \int_{{}_{(0)}\mathscr{H}} \xi^{3}_{(0)}\mathbf{g}^{\alpha}\cdot{}_{(0)}\boldsymbol{\sigma}\,\mathrm{d}\xi^{3},\\ {}_{(-1)}\tilde{\mathbf{m}}^{\alpha} &:= \int_{{}_{(-1)}\mathscr{H}} (\xi^{3}+{}_{(-1)}Z){}_{(-1)}\mathbf{g}^{\alpha}\cdot{}_{(-1)}\boldsymbol{\sigma}\,\mathrm{d}\xi^{3}, \end{aligned} \tag{131}$$

$${}_{(\ell)}\ell := \int_{{}_{(\ell)}\mathscr{H}} {}_{(\ell)}\mathbf{g}^{3}\cdot{}_{(\ell)}\boldsymbol{\sigma}\,\mathrm{d}\xi^{3}, \quad \text{for } \ell = -1,0,1. \tag{132}$$

The consequences of the assumptions (ii) and (iii) above are

$${}_{(1)}\tilde{\mathbf{m}}^{\alpha} = {}_{(-1)}\tilde{\mathbf{m}}^{\alpha} \quad \text{and} \quad {}_{(1)}\mathbf{n}^{\alpha} = -{}_{(-1)}\mathbf{n}^{\alpha}. \tag{133}$$

The consequences of the assumptions (ii) and (iii) above are also eqn (121) and

$${}_{(1)}\ddot{\mathbf{t}} = {}_{(-1)}\ddot{\mathbf{t}}. \tag{134}$$

Using eqns (121) and (134), the balance of linear momentum equations (122) becomes

$$\hat{\mathbf{n}}^{\alpha}{}_{,\alpha}+\mathbf{n}^{*} = \hat{A}^{0}_{\rho}\ddot{\mathbf{u}}. \tag{135}$$

The balance of total angular momentum equations (126) becomes

$$[\hat{\tilde{\mathbf{m}}}^{\alpha}+({}_{(0)}h+{}_{(1)}h)({}_{(1)}\mathbf{n}^{\alpha}-{}_{(-1)}\mathbf{n}^{\alpha})]_{,\alpha}-\ell+\hat{\tilde{\mathbf{m}}}^{*} = \sum_{\ell=-1}^{1} {}_{(\ell)}A^{2}_{\rho}\,{}_{(\ell)}\ddot{\mathbf{t}}+2{}_{(0)}h_{(1)}A^{1}_{\rho}({}_{(0)}\ddot{\mathbf{t}}-{}_{(1)}\ddot{\mathbf{t}}). \tag{136}$$

The balance of angular momentum equations (124) becomes

$$
\begin{aligned}
({}_{(0)}\tilde{\mathbf{m}}^{\alpha} + {}_{(0)}h_{(1)}\mathbf{n}^{\alpha} - {}_{(0)}h_{(-1)}\mathbf{n}^{\alpha})_{,\alpha} - {}_{(0)}\ell + {}_{(0)}\tilde{\mathbf{m}}^{*} \\
= 2\,{}_{(0)}h_{(1)}A^{1}_{\rho\,(1)}\ddot{\mathbf{t}} + {}_{(0)}A^{2}_{\rho\,(0)}\ddot{\mathbf{t}} + 2({}_{(0)}h)^{2}\,{}_{(1)}A^{0}_{\rho}({}_{(0)}\ddot{\mathbf{t}} - {}_{(1)}\ddot{\mathbf{t}}). \quad (137)
\end{aligned}
$$

Due to eqn (133), the balance of angular momentum eqn (127) vanishes identically. Thus the three nonlinear equations governing the motion of the symmetric sandwich plate with identical outer layer rotations are eqns (135)–(137).

The force boundary conditions corresponding to eqns (135)–(137) are obtained from $(85)_1$, (116) and $(85)_3$, respectively,

$$\hat{\mathbf{n}}^{\alpha} = \mathbf{n}^{*\alpha}, \quad (138)$$

$$\hat{\tilde{\mathbf{m}}}^{\alpha} + ({}_{(0)}h + {}_{(1)}h)({}_{(1)}\mathbf{n}^{\alpha} - {}_{(-1)}\mathbf{n}^{\alpha}) = \hat{\tilde{\mathbf{m}}}^{*}, \quad (139)$$

$${}_{(0)}\tilde{\mathbf{m}}^{\alpha} + {}_{(0)}h_{(1)}\mathbf{n}^{\alpha} - {}_{(0)}h_{(-1)}\mathbf{n}^{\alpha} = \tilde{\mathbf{m}}^{*\alpha}. \quad (140)$$

Now, decompose the displacement $\mathbf{u}$ and the director ${}_{(\ell)}\mathbf{t}$ in the material Cartesian coordinate as follows

$$\mathbf{u} := u\mathbf{E}^{1} + v\mathbf{E}^{2} + w\mathbf{E}^{3}, \quad (141)$$

$${}_{(\ell)}\mathbf{t} := {}_{(\ell)}\psi\mathbf{E}^{1} + {}_{(\ell)}\varphi\mathbf{E}^{2} + ({}_{(\ell)}\beta + 1)\mathbf{E}^{3}, \quad (142)$$

where $\{\mathbf{E}^{i}\}$ is the Cartesian basis in the reference configuration. From eqns (141) and (142), we obtain the second time rate of the displacement $\mathbf{u}$ and the director ${}_{(\ell)}\mathbf{t}$ as follows

$$\ddot{\mathbf{u}} := \ddot{u}\mathbf{E}^{1} + \ddot{v}\mathbf{E}^{2} + \ddot{w}\mathbf{E}^{3}, \quad (143)$$

$${}_{(\ell)}\ddot{\mathbf{t}} := {}_{(\ell)}\ddot{\psi}\mathbf{E}^{1} + {}_{(\ell)}\ddot{\varphi}\mathbf{E}^{2} + {}_{(\ell)}\ddot{\beta}\mathbf{E}^{3}. \quad (144)$$

Using eqns (143) and (144), we can easily verify that the component form of the equations of motion (135)–(137) are the same as equation (9) in Yu (1959). To simplify the presentation, we now just compare some component equations here.

Shown in Fig. 3 is the configuration of a symmetric sandwich plate and its corresponding geometric quantities written in the convention as in Yu (1959). The material properties in the present paper can be expressed as

$$
\begin{aligned}
&{}_{(1)}\rho = {}_{(-1)}\rho \Rightarrow \rho_2, \quad {}_{(1)}A^{1}_{\rho} = -{}_{(-1)}A^{1}_{\rho} \Rightarrow \tfrac{1}{2}\rho_2(h^2 - h_1^2), \\
&{}_{(0)}A^{1}_{\rho} = 0, \quad {}_{(0)}A^{0}_{\rho} \Rightarrow 2\rho_1 h_1, \quad {}_{(1)}A^{0}_{\rho} = {}_{(-1)}A^{0}_{\rho} \Rightarrow \rho_2 h_2, \quad {}_{(0)}A^{2}_{\rho} \Rightarrow \tfrac{2}{3}\rho_1 h_1^3. \quad (145)
\end{aligned}
$$

We now want to compare the first equation in eqns (135), (136) and (137) with eqns

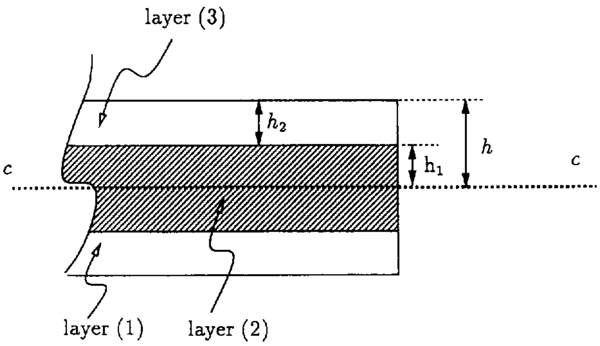

Fig. 3. Symmetric sandwich shell in Y. Y. Yu's convention: profile and geometric quantities.

$(9)_1$, $(9)_4$ and $(9)_7$ in Yu (1959), respectively. To this end, we list the equations $(9)_1$, $(9)_4$ and $(9)_7$ in Yu (1959) below.

$$\frac{N_x}{\partial x}+\frac{N_{xy}}{\partial y}+[(\tau_{zx3})_h-(\tau_{zx2})_{-h}] = 2(\rho_1 h_1+\rho_2 h_2)\ddot{\mathbf{u}}$$

$$\underbrace{\frac{\partial}{\partial x}\left[M_x+\left(h_1+\frac{h_2}{2}\right)(N_{x3}-N_{x2})\right]+\frac{\partial}{\partial y}\left[M_{xy}+\left(h_1+\frac{h_2}{2}\right)(N_{xy3}-N_{xy2})\right]}_{[1]}$$

$$-Q_x+\underbrace{h[(\tau_{zx3})_h-(\tau_{zx2})_{-h}]}_{[2]}$$

$$=\underbrace{\rho_1\frac{2h_1^3}{3}\ddot{\psi}_1+\rho_2 h_1(h^2-h_1^2)(\ddot{\psi}_1-\ddot{\psi}_2)+\rho_2\frac{2}{3}(h^3-h_1^3)\ddot{\psi}_2}_{[3]}$$

$$\frac{\partial}{\partial x}[M_{x1}+h_1(N_{x3}-N_{x2})]+\frac{\partial}{\partial y}[M_{xy1}+h_1(N_{xy3}-N_{xy2})]-Q_{x1}+h_1[(\tau_{zx3})_h-(\tau_{zx2})_{-h}]$$

$$=\rho_1\frac{2h_1^3}{3}\ddot{\psi}_1+2\rho_2 h_1^2 h_2(\ddot{\psi}_1-\ddot{\psi}_2)+\rho_2 h_1(h^2-h_1^2)\ddot{\psi}_2$$

Firstly, let us compare the first component of eqn (135) with eqn $(9)_1$ in Yu (1959). The first equation of eqn (135) can be written as

$$\hat{n}^{11}{}_{,1}+\hat{n}^{12}{}_{,2}+n^{*1} = \hat{A}^0_\rho \ddot{u}, \tag{146}$$

where the term $\hat{n}^{11}{}_{,1}+\hat{n}^{12}{}_{,2}$ corresponds to $N_x/\partial x+N_{xy}/\partial y$ in Yu (1959) and the assigned force n^{*1} corresponds to $(\tau_{zx3})_h-(\tau_{zx2})_{-h}$ in Yu (1959). Using eqn (145), the inertia force in (135) can be written as

$$\hat{A}^0_\rho \ddot{\mathbf{u}} \Rightarrow 2(\rho_1 h_1+\rho_2 h_2)\ddot{\mathbf{u}}. \tag{147}$$

Thus, eqn $(135)_1$ is exactly the same as eqn $(9)_1$ in Yu (1959).

Secondly, we compare the equation of total balance of angular moment, eqn (136), with eqn $(9)_4$ in Yu (1959). Note that

$${}_{(0)}\tilde{\mathbf{m}}^\alpha \Rightarrow \mathbf{M}_{\alpha 1},\quad {}_{(0)}h^+ \Rightarrow h_1,\quad {}_{(0)}h^- = {}_{(1)}h^+ \Rightarrow \frac{h_2}{2}. \tag{148}$$

The first square bracket in eqn (136) is the same as the term [1] in Yu (1959). Also notice that $\hat{\ell}$ has the same meaning as Q_x, and the boundary term $\hat{\mathbf{m}}^*$ is the same as term [2] in Yu (1959). Finally, by using assumption (iii), eqns (144) and (145), the inertia force in eqn (136) (left hand term) can reduce to term [2] in Yu (1959). Thus, the first component equation of eqn (136) is identical to eqn $(9)_4$ in Yu (1959).

Using the same procedure, we can easily prove that the first component of balance of angular momentum, eqn (137), can reduce to eqn $(9)_7$ in Yu (1959). The other component equations can also be proved in the same way.

5. STRAIN MEASURES AND CONSTITUTIVE LAWS

In this section, we derive the elastic constitutive equations in terms of the stress resultants and conjugate strain measures. To this end, we first introduce the layer effective stress resultants (resultant force ${}_{(\ell)}\tilde{n}^{\beta\alpha}$, and ${}_{(\ell)}\tilde{q}^\alpha$, and resultant couple ${}_{(\ell)}\tilde{m}^{\beta\alpha}$), and then derive their conjugate strain measures. The constitutive laws related to these quantities will be postulated. The reader is referred to Simo and Fox (1989) for further detail on the single layer case.

5.1. *Constitutive restriction*

In Section 3, we define the resultant force ${}_{(\ell)}\mathbf{n}^\alpha$ in eqn (39), the resultant couple ${}_{(\ell)}\tilde{\mathbf{m}}^\alpha$ in eqn (40) and the resultant director couple ${}_{(\ell)}\ell$ in eqn (41). These quantities are not independent, but are related to each other by a relation, called the constitutive restriction, arise from the local balance of angular momentum in the three-dimensional continuum, which in turn give rise to the symmetry of the Cauchy stress tensor. One can think of this constitutive restriction as the resultant form of the symmetry of the Cauchy stress tensor.

The local balance of angular momentum is written as

$$ {}_{(\ell)}\boldsymbol{\sigma} = {}_{(\ell)}\boldsymbol{\sigma}^t, \quad \text{or} \quad ({}_{(\ell)}\mathbf{g}^l \cdot {}_{(\ell)}\boldsymbol{\sigma}) \times {}_{(\ell)}\mathbf{g}_l = 0. \tag{149} $$

Multiply eqn $(149)_2$ with ${}_{(\ell)}j_t$, integrate the result over the layer thickness ${}_{(\ell)}\mathscr{H}$, and make use of eqn (21), we obtain

$$ \int_{(1)\mathscr{H}} \{({}_{(1)}\mathbf{g}^\alpha \cdot {}_{(1)}\boldsymbol{\sigma}) \times [{}_{(1)}\boldsymbol{\varphi}_{,\alpha} + (\xi^3 - {}_{(1)}Z)_{(1)}\mathbf{t}_{,\alpha}] + ({}_{(1)}\mathbf{g}^3 \cdot {}_{(1)}\boldsymbol{\sigma}) \times {}_{(1)}\mathbf{t}\}_{(1)}j_t \,\mathrm{d}\xi^3 = 0, $$

$$ \int_{(0)\mathscr{H}} \{({}_{(0)}\mathbf{g}^\alpha \cdot {}_{(0)}\boldsymbol{\sigma}) \times [{}_{(0)}\boldsymbol{\varphi}_{,\alpha} + \xi^3{}_{(0)}\mathbf{t}_{,\alpha}] + ({}_{(0)}\mathbf{g}^3 \cdot {}_{(0)}\boldsymbol{\sigma}) \times {}_{(0)}\mathbf{t}\}_{(0)}j_t \,\mathrm{d}\xi^3 = 0, $$

$$ \int_{(-1)\mathscr{H}} \{({}_{(-1)}\mathbf{g}^\alpha \cdot {}_{(-1)}\boldsymbol{\sigma}) \times [{}_{(-1)}\boldsymbol{\varphi}_{,\alpha} + (\xi^3 + {}_{(-1)}Z)_{(-1)}\mathbf{t}_{,\alpha}] + ({}_{(-1)}\mathbf{g}^3 \cdot {}_{(-1)}\boldsymbol{\sigma}) \times {}_{(-1)}\mathbf{t}\}_{(-1)}j_t \,\mathrm{d}\xi^3 = 0. \tag{150} $$

Substitute in equation (150) the definition of the resultant force ${}_{(\ell)}\mathbf{n}^\alpha$ in eqn (39), the resultant couple ${}_{(\ell)}\tilde{\mathbf{m}}^\alpha$ in eqn (40), and the director couple ${}_{(\ell)}\ell$ in eqn (41), we obtain the following relation linking these stress resultants:

$$ {}_{(\ell)}\mathbf{n}^\alpha \times {}_{(\ell)}\boldsymbol{\varphi}_{,\alpha} + {}_{(\ell)}\tilde{\mathbf{m}}^\alpha \times {}_{(\ell)}\mathbf{t}_{,\alpha} + {}_{(\ell)}\ell \times {}_{(\ell)}\mathbf{t} = 0, \tag{151} $$

which represents the restriction placed on the admissible form of the constitutive laws by the balance of angular momentum.

5.2. *Alternative form for the balance of angular momentum*

Using the constitutive restriction, eqn (151), the equations of motion for the balance of angular momentum $(84)_2$–$(84)_4$ can be recast in an alternative form. Define

$$ {}_{(\ell)}\mathbf{m}^\alpha = {}_{(\ell)}\mathbf{t} \times {}_{(\ell)}\tilde{\mathbf{m}}^\alpha, \quad {}_{(\ell)}\mathbf{m}^* = {}_{(\ell)}\mathbf{t} \times {}_{(\ell)}\tilde{\mathbf{m}}^*. \tag{152} $$

First, by forming the vector product of ${}_{(-1)}\mathbf{t}$ and $(84)_4$, we obtain

$$ {}_{(-1)}\mathbf{t} \times ({}_{(-1)}\tilde{\mathbf{m}}^\alpha - {}_{(-1)}h^+{}_{(-1)}\mathbf{n}^\alpha)_{,\alpha} - {}_{(-1)}\mathbf{t} \times {}_{(-1)}\ell + {}_{(-1)}\mathbf{t} \times {}_{(-1)}\tilde{\mathbf{m}}^* = {}_{(-1)}\mathbf{t} \times {}_{(-1)}\mathbf{C}. \tag{153} $$

By substituting eqn (151) in (153) and reorganizing the term in the parenthesis, notice that ${}_{(\ell)}h^-$ and ${}_{(\ell)}h^-$ are not functions of ξ^α, we obtain

$$({}_{(-1)}\mathbf{t}\times{}_{(-1)}\tilde{\mathbf{m}}^{\alpha})_{,\alpha}-{}_{(-1)}\mathbf{t}_{,\alpha}\times{}_{(-1)}\tilde{\mathbf{m}}^{\alpha}-{}_{(-1)}h^{+}\,{}_{(-1)}\mathbf{t}\times{}_{(-1)}\mathbf{n}^{\alpha}_{,\alpha}+{}_{(-1)}\boldsymbol{\varphi}_{,\alpha}\times{}_{(-1)}\mathbf{n}^{\alpha}$$
$$+{}_{(-1)}\mathbf{t}_{,\alpha}\times{}_{(-1)}\tilde{\mathbf{m}}^{\alpha}+{}_{(-1)}\mathbf{t}\times{}_{(-1)}\tilde{\mathbf{m}}^{*}={}_{(-1)}\mathbf{t}\times{}_{(-1)}\mathbf{C}. \quad (154)$$

Using eqn (152), eqn (154) can be rewritten as

$${}_{(-1)}\mathbf{m}^{\alpha}_{,\alpha}-{}_{(-1)}h^{+}\,{}_{(-1)}\mathbf{t}\times{}_{(-1)}\mathbf{n}^{\alpha}_{,\alpha}+{}_{(-1)}\boldsymbol{\varphi}_{,\alpha}\times{}_{(-1)}\mathbf{n}^{\alpha}+{}_{(-1)}\mathbf{m}^{*}={}_{(-1)}\mathbf{t}\times{}_{(-1)}\mathbf{C}. \quad (155)$$

Similarly, we obtain from eqns $(84)_3$ and $(84)_2$ the following alternative equations of motion for angular momentum

$${}_{(0)}\mathbf{m}^{\alpha}_{,\alpha}+{}_{(0)}h^{-}\,{}_{(0)}\mathbf{t}\times{}_{(1)}\mathbf{n}^{\alpha}_{,\alpha}-{}_{(0)}h^{-}\,{}_{(0)}\mathbf{t}\times{}_{(-1)}\mathbf{n}^{\alpha}_{,\alpha}+{}_{(0)}\boldsymbol{\varphi}_{,\alpha}\times{}_{(0)}\mathbf{n}^{\alpha}+{}_{(0)}\mathbf{m}^{*}={}_{(0)}\mathbf{t}\times{}_{(0)}\mathbf{C}. \quad (156)$$

$${}_{(1)}\mathbf{m}^{\alpha}_{,\alpha}+\underbrace{{}_{(1)}h^{-}\,{}_{(1)}\mathbf{t}\times{}_{(1)}\mathbf{n}^{\alpha}_{,\alpha}+{}_{(1)}\boldsymbol{\varphi}_{,\alpha}\times{}_{(1)}\mathbf{n}^{\alpha}}_{}+{}_{(1)}\mathbf{m}^{*}={}_{(1)}\mathbf{t}\times{}_{(1)}\mathbf{C}. \quad (157)$$

Remark 5.1. We now proceed to prove that eqns (155)–(157) can be reduced to eqns (3.37)–(3.39) in Vu-Quoc and Ebcioğlu (1995) for the two-dimensional geometrically-exact sandwich beams. To simplify the verification, we only consider eqn (157) for the top layer (1), which corresponds to eqn (3.39) in Vu-Quoc and Ebcioğlu (1995), as an example. To lead the readers through this verification, we recall that eqn (3.39) in Vu-Quoc and Ebcioğlu (1995) reads as follows

$${}_{(1)}m_{,S}+\underbrace{[{}_{(0)}\boldsymbol{\Phi}_{0,S}\times{}_{(1)}\mathbf{n}]\cdot\mathbf{e}_3-{}_{(0)}h_{(0)}\theta_{,S}({}_{(1)}\mathbf{n}\cdot{}_{(0)}\mathbf{t}_2)}_{}\,\underbrace{-{}_{(1)}h({}_{(1)}\mathbf{n}\cdot{}_{(1)}\mathbf{t}_1)_{,S}}_{}+{}_{(1)}\bar{\mathscr{M}}={}_{(1)}C, \quad (158)$$

where the layer indices have been changed to be consistent with this paper (see Remark 2.1). In eqn (158), ${}_{(1)}m_{,S}$, ${}_{(1)}\mathbf{n}$, ${}_{(1)}\bar{\mathscr{M}}$ and ${}_{(1)}C$ are, respectively, the resultant contact couple, the resultant contact force, the resultant assigned couple and the inertia couple associated with layer (1), whereas ${}_{(0)}\boldsymbol{\Phi}_0$, ${}_{(0)}\theta$, ${}_{(\ell)}h$ and ${}_{(\ell)}\mathbf{t}_{\alpha}$ are, respectively, the deformation map of the centroidal line of the core layer, the rotation of the centroidal line of the core layer, the half-layer thickness of layer (ℓ), and the basis vector attached to the layer cross-section. The subscript $,S$ denotes the derivative with respect to the spatial coordinate S along the beam axis. By making the projection of eqn (157) on the spatial Cartesian basis vector $\mathbf{e}_3$, it is easy to verify that the quantities ${}_{(1)}\mathbf{m}^{\alpha}_{,\alpha}$, ${}_{(1)}\mathbf{m}^{*}$ and ${}_{(1)}\mathbf{t}\times{}_{(1)}\mathbf{C}$ for shells in eqn (157) (i.e. the non-underbraced terms) correspond to the quantities ${}_{(1)}m_{,S}$, ${}_{(1)}\bar{\mathscr{M}}$ and ${}_{(1)}C$ for beams in eqn (158), respectively. Using eqn (9)—i.e. the relation between deformation map of the neutral surface of layer (1) and that of the core layer (0)—the underbraced terms in eqn (157) can be reorganized as follows

$${}_{(1)}h^{-}\,{}_{(1)}\mathbf{t}\times{}_{(1)}\mathbf{n}^{\alpha}_{,\alpha}+{}_{(1)}\boldsymbol{\varphi}_{,\alpha}\times{}_{(1)}\mathbf{n}^{\alpha}={}_{(0)}\boldsymbol{\varphi}_{,\alpha}\times{}_{(1)}\mathbf{n}^{\alpha}_{,\alpha}+{}_{(0)}h^{+}\,{}_{(0)}\mathbf{t}_{,\alpha}\times{}_{(1)}\mathbf{n}^{\alpha}_{,\alpha}+{}_{(1)}h^{-}({}_{(1)}\mathbf{t}\times{}_{(1)}\mathbf{n}^{\alpha})_{,\alpha}. \quad (159)$$

Notice that the director ${}_{(0)}\mathbf{t}$ in shell theory corresponds to the basis vector ${}_{(0)}\mathbf{t}_2$ in beam theory, and that in the plane beams we have ${}_{(0)}\mathbf{t}_{2,S}=-{}_{(0)}\theta_{,S}\,{}_{(0)}\mathbf{t}_1$ and ${}_{(0)}\mathbf{t}_1\times{}_{(0)}\mathbf{t}_2=\mathbf{e}_3$. Thus, by forming the projection of eqn (159) on $\mathbf{e}_3$, the right-hand-side terms in eqn (159) reduce to the underbraced terms in eqn (158).

We thus proved that eqn (159) can be reduced to (3.39) in Vu-Quoc and Ebcioğlu (1995). Similarly, we can prove that the angular momentum equations (155), (156) for sandwich shells can be reduced to the angular momentum equations (3.37), (3.38) for sandwich beams in Vu-Quoc and Ebcioğlu (1995). The computational formulation for the equations of motion in Vu-Quoc and Ebcioğlu (1995) and several numerical simulations are discussed in Vu-Quoc and Deng (1995c) and Vu-Quoc and Deng (1995a). ■

5.3. *Effective stress resultants*

To obtain properly the invariant elastic constitutive laws for geometrically-exact sandwich shell, we make use of the definition of the effective stress resultants ${}_{(\ell)}\tilde{\mathbf{n}}^{\beta\alpha}$ and ${}_{(\ell)}\tilde{\mathbf{q}}^{\alpha}$ (see Simo and Fox (1989)). To this end, we must make use of the inextensibility of the director, i.e. ${}_{(\ell)}\mathbf{t} \in S^2$.

First, we decompose the spatial derivative ${}_{(\ell)}\mathbf{t}_{,\alpha}$ of the director ${}_{(\ell)}\mathbf{t}$ along the neutral surface convected basis $\{{}_{(\ell)}\boldsymbol{\varphi}_{,\alpha}, {}_{(\ell)}\mathbf{t}\}$ as

$$ {}_{(\ell)}\mathbf{t}_{,\alpha} = {}_{(\ell)}\lambda^{\mu}_{.\alpha}\,{}_{(\ell)}\boldsymbol{\varphi}_{,\mu} + {}_{(\ell)}\lambda^{3}_{.\alpha}\,{}_{(\ell)}\mathbf{t}. \tag{160} $$

Using the constraint ${}_{(\ell)}\mathbf{t} \in S^2$, or equivalently ${}_{(\ell)}\mathbf{t}\cdot{}_{(\ell)}\mathbf{t} = 1$, we obtain

$$ {}_{(\ell)}\mathbf{t}_{,\alpha}\cdot{}_{(\ell)}\mathbf{t} = 0, \quad {}_{(\ell)}\dot{\mathbf{t}}\cdot{}_{(\ell)}\mathbf{t} = 0, \quad {}_{(\ell)}\mathbf{t}\cdot{}_{(\ell)}\dot{\mathbf{t}}_{,\alpha} = -{}_{(\ell)}\mathbf{t}_{,\alpha}\cdot{}_{(\ell)}\dot{\mathbf{t}}, \quad \text{for } \ell = 1,2,3. \tag{161} $$

Using eqn $(161)_1$, we obtain from eqn (160) the expression for ${}_{(\ell)}\lambda^{3}_{.\alpha}$ as

$$ {}_{(\ell)}\lambda^{3}_{.\alpha} = -{}_{(\ell)}\lambda^{\mu}_{.\alpha}\,{}_{(\ell)}\boldsymbol{\varphi}_{,\mu}\cdot{}_{(\ell)}\mathbf{t}. \tag{162} $$

We now resolve the stress resultants ${}_{(\ell)}\mathbf{n}^{\alpha}$ and ${}_{(\ell)}\tilde{\mathbf{m}}^{\alpha}$ along the neutral surface convected basis $\{{}_{(\ell)}\boldsymbol{\varphi}_{,\alpha}, {}_{(\ell)}\mathbf{t}\}$ as

$$ {}_{(\ell)}\mathbf{n}^{\alpha} = {}_{(\ell)}n^{\alpha\beta}\,{}_{(\ell)}\boldsymbol{\varphi}_{,\beta} + {}_{(\ell)}q^{\alpha}\,{}_{(\ell)}\mathbf{t}, \tag{163} $$

$$ {}_{(\ell)}\tilde{\mathbf{m}}^{\alpha} = {}_{(\ell)}\tilde{m}^{\alpha\beta}\,{}_{(\ell)}\boldsymbol{\varphi}_{,\beta} + {}_{(\ell)}\tilde{m}^{\alpha 3}\,{}_{(\ell)}\mathbf{t}. \tag{164} $$

Substituting eqns (163) and (164) into the constitutive restriction, eqn (151) we obtain

$$ ({}_{(\ell)}n^{\alpha\beta}\,{}_{(\ell)}\boldsymbol{\varphi}_{,\beta})\times{}_{(\ell)}\boldsymbol{\varphi}_{,\alpha} + ({}_{(\ell)}q^{\alpha}\,{}_{(\ell)}\mathbf{t})\times{}_{(\ell)}\boldsymbol{\varphi}_{,\alpha} + ({}_{(\ell)}\tilde{m}^{\alpha\beta}\,{}_{(\ell)}\boldsymbol{\varphi}_{,\beta} + {}_{(\ell)}\tilde{m}^{\alpha 3}\,{}_{(\ell)}\mathbf{t})\times{}_{(\ell)}\mathbf{t}_{,\alpha} + {}_{(\ell)}\boldsymbol{\ell}\times{}_{(\ell)}\mathbf{t} = 0. \tag{165} $$

The projection of eqn (165) on ${}_{(\ell)}\mathbf{t}$ yields

$$ [({}_{(\ell)}n^{\alpha\beta}\,{}_{(\ell)}\boldsymbol{\varphi}_{,\beta})\times{}_{(\ell)}\boldsymbol{\varphi}_{,\alpha}]\cdot{}_{(\ell)}\mathbf{t} + [{}_{(\ell)}\tilde{m}^{\alpha\beta}\,{}_{(\ell)}\boldsymbol{\varphi}_{,\beta})\times{}_{(\ell)}\mathbf{t}_{,\alpha}]\cdot{}_{(\ell)}\mathbf{t} = 0, \tag{166} $$

which, via the use of eqn (160), leads to

$$ ({}_{(\ell)}n^{\alpha\beta} + {}_{(\ell)}\lambda^{\alpha}_{.\mu}\,{}_{(\ell)}\tilde{m}^{\mu\beta})({}_{(\ell)}\boldsymbol{\varphi}_{,\beta}\times{}_{(\ell)}\boldsymbol{\varphi}_{,\alpha})\cdot{}_{(\ell)}\mathbf{t} = 0. \tag{167} $$

Furthermore, by interchanging the dummy summation variables, we obtain another form of eqn (167)

$$ ({}_{(\ell)}n^{\alpha\beta} - {}_{(\ell)}\lambda^{\beta}_{.\mu}\,{}_{(\ell)}\tilde{m}^{\mu\alpha})({}_{(\ell)}\boldsymbol{\varphi}_{,\beta}\times{}_{(\ell)}\boldsymbol{\varphi}_{,\alpha})\cdot{}_{(\ell)}\mathbf{t} = 0. \tag{168} $$

Consequently, from eqns (167) or (168), we define the effective resultant membrane stress

$$ {}_{(\ell)}\tilde{n}^{\alpha\beta} := {}_{(\ell)}n^{\alpha\beta} + {}_{(\ell)}\lambda^{\alpha}_{.\mu}\,{}_{(\ell)}\tilde{m}^{\mu\beta} = {}_{(\ell)}n^{\alpha\beta} - {}_{(\ell)}\lambda^{\beta}_{.\mu}\,{}_{(\ell)}\tilde{m}^{\mu\alpha}. \tag{169} $$

Since $({}_{(\ell)}\boldsymbol{\varphi}_{,\beta}\times{}_{(\ell)}\boldsymbol{\varphi}_{,\alpha})\cdot{}_{(\ell)}\mathbf{t} \neq 0$ for $\alpha \neq \beta$, it follows from eqns (167) and (168) that

$$ {}_{(\ell)}\tilde{n}^{\alpha\beta} = {}_{(\ell)}\tilde{n}^{\beta\alpha}, \tag{170} $$

i.e. the effective membrane force tensor ${}_{(\ell)}\tilde{\mathbf{n}} = {}_{(\ell)}\tilde{n}^{\alpha\beta}\,{}_{(\ell)}\boldsymbol{\varphi}_{,\alpha} \otimes {}_{(\ell)}\boldsymbol{\varphi}_{,\beta}$ is symmetric.

Remark 5.2. We give eqn (169) two possible expressions for the effective membrane for ${}_{(\ell)}\tilde{n}^{\alpha\beta}$. In Simo and Fox (1989), only expression $(169)_2$ was obtained. It should also be noted that the convention for the superscripts of the components of the stress tensors employed here follows consistently the convention used in Malvern (1969), and differs from the convention used in Simo and Fox (1989); see the definitions (39)–(41). ■

The components of the director couple ${}_{(\ell)}\ell$ with respect to the basis vectors $\{{}_{(\ell)}\boldsymbol{\varphi}_{,\alpha}\,{}_{(\ell)}\mathbf{t}\}$ can be expressed in terms of the stress resultants as follows. Let

$$ {}_{(\ell)}\ell = {}_{(\ell)}\mathbf{L}^{\alpha}\,{}_{(\ell)}\boldsymbol{\varphi}_{,\alpha} + {}_{(\ell)}\lambda_{(\ell)}\mathbf{t}. \tag{171} $$

Forming the dot product of the constitutive restriction (165) with the basis vector ${}_{(\ell)}\boldsymbol{\varphi}_{,\gamma}$ and making use of eqn (160), we obtain explicitly the expression for the components ${}_{(\ell)}\mathbf{L}^{\alpha}$ as

$$ {}_{(\ell)}\mathbf{L}^{\alpha} = {}_{(\ell)}q^{\alpha} - {}_{(\ell)}\lambda^{3}_{\cdot\mu}\,{}_{(\ell)}\tilde{m}^{\mu\alpha} + {}_{(\ell)}\lambda^{\alpha}_{\cdot\mu}\,{}_{(\ell)}\tilde{m}^{\mu 3}, \tag{172} $$

in terms of the stress resultant, and arrive at the following expression for the director couple ${}_{(\ell)}\ell$

$$ {}_{(\ell)}\ell = ({}_{(\ell)}q^{\alpha} - {}_{(\ell)}\lambda^{3}_{\cdot\mu}\,{}_{(\ell)}\tilde{m}^{\mu\alpha} + {}_{(\ell)}\lambda^{\alpha}_{\cdot\mu}\,{}_{(\ell)}\tilde{m}^{\mu 3})\,{}_{(\ell)}\boldsymbol{\varphi}_{,\alpha} + {}_{(\ell)}\lambda_{(\ell)}\mathbf{t}. \tag{173} $$

Let the effective resultant shear force ${}_{(\ell)}\tilde{q}^{\alpha}$ be defined as follows

$$ {}_{(\ell)}\tilde{q}^{\alpha} := {}_{(\ell)}q^{\alpha} - {}_{(\ell)}\lambda^{3}_{\cdot\mu}\,{}_{(\ell)}\tilde{m}^{\alpha\mu}, \tag{174} $$

then the director couple ${}_{(\ell)}\ell$ in eqn (173) is now expressed as

$$ {}_{(\ell)}\ell = {}_{(\ell)}\tilde{q}^{\alpha}\,{}_{(\ell)}\boldsymbol{\varphi}_{,\alpha} + {}_{(\ell)}\lambda^{\alpha}_{\cdot\mu}\,{}_{(\ell)}\tilde{m}^{\mu 3}\,{}_{(\ell)}\boldsymbol{\varphi}_{,\alpha} + {}_{(\ell)}\lambda_{(\ell)}\mathbf{t}. \tag{175} $$

The effective stress resultants are crucial in the development of the constitutive equations for sandwich shells in the following section.

5.4. *Reduced stress power and conjugate strain measures*

The convected basis vector at the spatial neutral surface are denoted by $\{{}_{(\ell)}\mathbf{a}_i\}$ and are obtained by evaluating the basis vectors $\{{}_{(\ell)}\mathbf{g}_i\}$ at $\xi^3 = 0$, i.e.

$$ \{{}_{(\ell)}\mathbf{a}_i\} := \{{}_{(\ell)}\mathbf{a}_1, {}_{(\ell)}\mathbf{a}_2, {}_{(\ell)}\mathbf{a}_3\} := \{{}_{(\ell)}\boldsymbol{\varphi}_{,1}, {}_{(\ell)}\boldsymbol{\varphi}_{,2}, {}_{(\ell)}\mathbf{t}\}|_{\xi^3=0}, \quad \text{for } \ell = -1, 0, 1. \tag{176} $$

The basis vectors $\{{}_{(\ell)}\mathbf{a}^i\}$, dual to $\{{}_{(\ell)}\mathbf{a}_i\}$, are defined by the standard orthogonal relation

$$ \langle {}_{(\ell)}\mathbf{a}_i, {}_{(\ell)}\mathbf{a}^j \rangle = \delta^j_i, \tag{177} $$

where δ^j_i is the Kronecker delta. The convected basis in the reference configuration is given by specifying $t = 0$ in the spatial-neutral-surface convected basis to obtain

$$ \{{}_{(\ell)}\mathbf{A}_I\} := \{{}_{(\ell)}\mathbf{A}_1, {}_{(\ell)}\mathbf{A}_2, {}_{(\ell)}\mathbf{A}_3\} := \{{}_{(\ell)}\mathbf{a}_1, {}_{(\ell)}\mathbf{a}_2, {}_{(\ell)}\mathbf{a}_3\}|_{t=0}, \quad \text{for } \ell = -1, 0, 1. \tag{178} $$

The basis $\{{}_{(\ell)}\mathbf{A}^J\}$, dual to the reference-neutral-surface convected basis $\{{}_{(\ell)}\mathbf{A}_I\}$, is defined similar to eqn (177).

The membrane strain ${}_{(\ell)}\boldsymbol{\varepsilon}$, the bending strain measure ${}_{(\ell)}\boldsymbol{\rho}$ and the transverse shear strain ${}_{(\ell)}\boldsymbol{\delta}$—which, as we will see later, are conjugate to the resultant membrane stress ${}_{(\ell)}\tilde{n}^{\beta\alpha}$, the resultant couple ${}_{(\ell)}\tilde{m}^{\beta\alpha}$ and the transverse shear ${}_{(\ell)}\tilde{q}^{\alpha}$, respectively—can be defined as follows

$$ {}_{(\ell)}\boldsymbol{\varepsilon} := {}_{(\ell)}\varepsilon_{\alpha\beta}\,{}_{(\ell)}\mathbf{a}^{\alpha}\otimes{}_{(\ell)}\mathbf{a}^{\beta}, \tag{179} $$

$$ {}_{(\ell)}\boldsymbol{\rho} := {}_{(\ell)}\rho_{\alpha\beta}\,{}_{(\ell)}\mathbf{a}^{\alpha}\otimes{}_{(\ell)}\mathbf{a}^{\beta}, \tag{180} $$

$$ {}_{(\ell)}\boldsymbol{\delta} := {}_{(\ell)}\delta_{\alpha}\,{}_{(\ell)}\mathbf{a}^{\alpha}. \tag{181} $$

The components of the strain tensor ${}_{(\ell)}\boldsymbol{\varepsilon}$ are given by

$$ {}_{(\ell)}\varepsilon_{\alpha\beta} = \tfrac{1}{2}({}_{(\ell)}a_{\alpha\beta} - {}_{(\ell)}A_{\alpha\beta}), \tag{182} $$

where

$$ {}_{(\ell)}a_{\alpha\beta} := {}_{(\ell)}\mathbf{a}_{\alpha}\cdot{}_{(\ell)}\mathbf{a}_{\beta}, \quad {}_{(\ell)}A_{\alpha\beta} := {}_{(\ell)}\mathbf{A}_{\alpha}\cdot{}_{(\ell)}\mathbf{A}_{\beta} \tag{183} $$

are the component of the Riemann metric tensors of layer (ℓ), ${}_{(\ell)}a_{\alpha\beta}\,{}_{(\ell)}\mathbf{a}^{\alpha}\otimes{}_{(\ell)}\mathbf{a}^{\beta}$ and ${}_{(\ell)}A_{\alpha\beta}\,{}_{(\ell)}\mathbf{A}^{\alpha}\otimes{}_{(\ell)}\mathbf{A}^{\beta}$, in the spatial configuration and in the reference configuration, respectively [see, e.g. Naghdi (1972), Marsden and Hughes (1983)].

The components of the bending strain tensor ${}_{(\ell)}\boldsymbol{\rho}$ is given by

$$ {}_{(\ell)}\rho_{\alpha\beta} = {}_{(\ell)}\kappa_{\alpha\beta} - {}_{(\ell)}\kappa^{0}_{\alpha\beta}, \tag{184} $$

where the spatial director (non-symmetric) metric ${}_{(\ell)}\kappa_{\alpha\beta}$ for layer (ℓ) is defined as

$$ {}_{(\ell)}\kappa_{\alpha\beta} := {}_{(\ell)}\mathbf{a}_{\alpha}\cdot{}_{(\ell)}\mathbf{t}_{,\beta}, \tag{185} $$

and reference director (non-symmetric) metric ${}_{(\ell)}\kappa^{0}_{\alpha\beta}$ for layer (ℓ) is defined as

$$ {}_{(\ell)}\kappa^{0}_{\alpha\beta} := {}_{(\ell)}\mathbf{A}_{\alpha}\cdot{}_{(\ell)}\mathbf{t}_{0,\beta}. \tag{186} $$

The components of the transverse shear strain ${}_{(\ell)}\boldsymbol{\delta}_{\alpha}$ are given by

$$ {}_{(\ell)}\delta_{\alpha} \equiv {}_{(\ell)}\gamma_{\alpha} - {}_{(\ell)}\gamma^{0}_{\alpha}, \tag{187} $$

where

$$ {}_{(\ell)}\gamma_{\alpha} = {}_{(\ell)}\mathbf{a}_{\alpha}\cdot{}_{(\ell)}\mathbf{t}, \quad {}_{(\ell)}\gamma^{0}_{\alpha} = {}_{(\ell)}\mathbf{A}_{\alpha}\cdot{}_{(\ell)}\mathbf{t}_{0}, \tag{188} $$

which are measures of how much the director ${}_{(\ell)}\mathbf{t}$ and the director ${}_{(\ell)}\mathbf{t}_0$ depart from the normal to the neutral surfaces in the spatial configuration and in the reference configuration, respectively.

We have the following relations between the spatial basis ${}_{(\ell)}\mathbf{a}_{\beta}$ and their basis ${}_{(\ell)}\mathbf{a}^{\alpha}$:

$$ {}_{(\ell)}\mathbf{a}^{\alpha} = {}_{(\ell)}a^{\alpha\beta}\,{}_{(\ell)}\mathbf{a}_{\beta}, \quad {}_{(\ell)}a^{\alpha\beta} = {}_{(\ell)}\mathbf{a}^{\alpha}\cdot{}_{(\ell)}\mathbf{a}^{\alpha}, \quad {}_{(\ell)}a^{\alpha\gamma}\cdot{}_{(\ell)}a_{\beta\gamma} = \delta^{\alpha}_{\beta}. \tag{189} $$

From eqns (183), (185) and (188), we have the following time rate

$$ {}_{(\ell)}\dot{a}_{\alpha\beta} = {}_{(\ell)}\dot{\mathbf{a}}_{\alpha}\cdot{}_{(\ell)}\mathbf{a}_{\beta} + {}_{(\ell)}\mathbf{a}_{\alpha}\cdot{}_{(\ell)}\dot{\mathbf{a}}_{\beta}, \tag{190} $$

$$ {}_{(\ell)}\dot{\kappa}_{\alpha\beta} = {}_{(\ell)}\dot{\mathbf{a}}_{\alpha}\cdot{}_{(\ell)}\mathbf{t}_{,\beta} + {}_{(\ell)}\mathbf{a}_{\alpha}\cdot{}_{(\ell)}\dot{\mathbf{t}}_{,\beta}, \tag{191} $$

$$ {}_{(\ell)}\dot{\gamma}_{\alpha\beta} = {}_{(\ell)}\dot{\mathbf{a}}_{\alpha}\cdot{}_{(\ell)}\mathbf{t} + {}_{(\ell)}\mathbf{a}_{\alpha}\cdot{}_{(\ell)}\dot{\mathbf{t}}. \tag{192} $$

Relation (190)–(192) play an important role in the development of the expression of the contact stress power and to identify the conjugate strain measures.

To this end, substitute eqns (163), (164), and (175) into the integrand ${}_{(\ell)}I$ of the stress power $\mathscr{P}_{\mathrm{c}}$ in eqn (42) to arrive at

$$
\begin{aligned}
{}_{(\ell)}I &= {}_{(\ell)}\mathbf{n}^{\alpha} \cdot {}_{(\ell)}\dot{\boldsymbol{\varphi}}_{,\alpha} + {}_{(\ell)}\tilde{\mathbf{m}}^{\alpha} \cdot {}_{(\ell)}\dot{\mathbf{t}}_{,\alpha} + {}_{(\ell)}\boldsymbol{\ell} \cdot {}_{(\ell)}\dot{\mathbf{t}} \\
&= \underbrace{({}_{(\ell)}n^{\alpha\beta}\,{}_{(\ell)}\boldsymbol{\alpha}_{\beta} + {}_{(\ell)}q^{\alpha}\,{}_{(\ell)}\mathbf{t}) \cdot {}_{(\ell)}\dot{\mathbf{a}}_{\alpha}}_{[1]} + \Big(\underbrace{{}_{(\ell)}\tilde{m}^{\alpha\beta}\,{}_{(\ell)}\mathbf{a}_{\beta}}_{[2a]} + \underbrace{{}_{(\ell)}\tilde{m}^{\alpha 3}\,{}_{(\ell)}\mathbf{t}}_{[2b]}\Big) \cdot {}_{(\ell)}\dot{\mathbf{t}}_{,\alpha} \\
&\quad + \Big(\underbrace{{}_{(\ell)}\tilde{q}^{\alpha}\,{}_{(\ell)}\mathbf{a}_{\alpha} + {}_{(\ell)}\hat{\lambda}^{\alpha}_{.\mu}\,{}_{(\ell)}\tilde{m}^{\mu 3}\,{}_{(\ell)}\mathbf{a}_{\alpha}}_{[3b]} + \underbrace{{}_{(\ell)}\hat{\lambda}\,{}_{(\ell)}\mathbf{t}}_{[3b]}\Big) \cdot {}_{(\ell)}\dot{\mathbf{t}}. \qquad (193)
\end{aligned}
$$

Next, using eqns $(169)_2$ and (174) in part [1] of eqn (193); eqn (191) in part [2a] of (193), and $(161)_2$ in part [3b] of (193), we can rewrite ${}_{(\ell)}I$ as follows

$$
\begin{aligned}
{}_{(\ell)}I &= \Big(\underbrace{{}_{(\ell)}\tilde{n}^{\alpha\beta}}_{[1a]} + \underbrace{{}_{(\ell)}\hat{\lambda}^{\beta}_{.\mu}\,{}_{(\ell)}\tilde{m}^{\mu\alpha}}_{[1b]}\Big)\,{}_{(\ell)}\mathbf{a}_{\beta} \cdot {}_{(\ell)}\dot{\mathbf{a}}_{\alpha} + \Big(\underbrace{{}_{(\ell)}\tilde{q}^{\alpha}}_{[2a]} + \underbrace{{}_{(\ell)}\hat{\lambda}^{3}_{.\mu}\,{}_{(\ell)}\tilde{m}^{\mu\alpha}}_{[2b]}\Big)\,{}_{(\ell)}\mathbf{t} \cdot {}_{(\ell)}\dot{\mathbf{a}}_{\alpha} \\
&\quad + {}_{(\ell)}\tilde{m}^{\alpha\beta}\Big({}_{(\ell)}\dot{\kappa}_{\alpha\beta} \underbrace{- {}_{(\ell)}\dot{\mathbf{a}}_{\beta} \cdot {}_{(\ell)}\mathbf{t}_{,\alpha}}_{[3]}\Big) - \underbrace{{}_{(\ell)}\tilde{m}^{\alpha 3}\,{}_{(\ell)}\mathbf{t}_{,\alpha} \cdot {}_{(\ell)}\dot{\mathbf{t}}}_{[4]} \\
&\quad + \Big(\underbrace{{}_{(\ell)}\tilde{q}^{\alpha}}_{[5a]} + \underbrace{{}_{(\ell)}\hat{\lambda}^{\alpha}_{.\mu}\,{}_{(\ell)}\tilde{m}^{\mu 3}}_{[5b]}\Big)\,{}_{(\ell)}\mathbf{a}_{\alpha} \cdot {}_{(\ell)}\dot{\mathbf{t}}. \qquad (194)
\end{aligned}
$$

By the symmetry of ${}_{(\ell)}n^{\alpha\beta}$ as expressed in eqn (170) and by definition (190) of ${}_{(\ell)}\dot{a}_{\alpha\beta}$, part [1a] of eqn (194) can be written as

$$
{}_{(\ell)}\tilde{n}^{\alpha\beta}\,{}_{(\ell)}\mathbf{a}_{\beta} \cdot {}_{(\ell)}\dot{\mathbf{a}}_{\alpha} = \tfrac{1}{2}\,{}_{(\ell)}\tilde{n}^{\alpha\beta}[{}_{(\ell)}\dot{\mathbf{a}}_{\alpha} \cdot {}_{(\ell)}\mathbf{a}_{\beta} + {}_{(\ell)}\mathbf{a}_{\alpha} \cdot {}_{(\ell)}\dot{\mathbf{a}}_{\beta}] = \tfrac{1}{2}\,{}_{(\ell)}\tilde{n}^{\alpha\beta}\,{}_{(\ell)}\dot{a}_{\alpha\beta}. \qquad (195)
$$

part [2a] and [5a] of eqn (194) can be combined to yield

$$
{}_{(\ell)}\tilde{q}^{\alpha}({}_{(\ell)}\dot{\mathbf{a}}_{\alpha} \cdot {}_{(\ell)}\mathbf{t} + {}_{(\ell)}\mathbf{t} \cdot {}_{(\ell)}\dot{\mathbf{a}}_{\alpha}) = {}_{(\ell)}\tilde{q}^{\alpha}\,{}_{(\ell)}\dot{\gamma}_{\alpha}, \qquad (196)
$$

by virtue of definition (192) of ${}_{(\ell)}\dot{\gamma}_{\alpha}$, part [1b] and [2b] of (194) are combined to yield

$$
{}_{(\ell)}\tilde{m}^{\mu\alpha}\,{}_{(\ell)}\dot{\mathbf{a}}_{\alpha} \cdot ({}_{(\ell)}\hat{\lambda}^{\beta}_{.\mu}\,{}_{(\ell)}\mathbf{a}_{\beta} + {}_{(\ell)}\hat{\lambda}^{3}_{.\mu}\,{}_{(\ell)}\mathbf{t}) = {}_{(\ell)}\tilde{m}^{\mu\alpha}\,{}_{(\ell)}\dot{\mathbf{a}}_{\alpha} \cdot {}_{(\ell)}\mathbf{t}_{,\mu}, \qquad (197)
$$

by virtue of (160). Clearly, eqn (197) cancels part [3] in eqn (194); thus parts [1b], [2b], and [3] in eqn (194) cancel each other. With the aid of eqns (160) and $(161)_2$, part [4] in eqn (194) can be written as

$$
{}_{(\ell)}\tilde{m}^{\alpha 3}\,{}_{(\ell)}\dot{\mathbf{t}} \cdot ({}_{(\ell)}\hat{\lambda}^{\mu}_{.\alpha}\,{}_{(\ell)}\mathbf{a}_{\mu} + {}_{(\ell)}\hat{\lambda}^{3}_{.\alpha}\,{}_{(\ell)}\mathbf{t}) = {}_{(\ell)}\hat{\lambda}^{\alpha}_{.\mu}\,{}_{(\ell)}\tilde{m}^{\mu 3}\,{}_{(\ell)}\mathbf{a}_{\alpha} \cdot {}_{(\ell)}\dot{\mathbf{t}}, \qquad (198)
$$

which cancels part [5b] of eqn (194); thus parts [4] and [5b] in eqn (194) cancel each other. Finally, using eqns (195) and (196) in expression (194) for ${}_{(\ell)}I$, the stress power $\mathscr{P}_{\mathrm{c}}$ in eqn (42) of geometrically-exact sandwich shells can now be written as

$$\mathscr{P}_c = \sum_{\ell=-1}^{1} \int_{\mathscr{A}} [\tfrac{1}{2}\,{}_{(\ell)}\tilde{n}^{\beta\alpha}\,{}_{(\ell)}\dot{a}_{\alpha\beta} + {}_{(\ell)}\tilde{m}^{\beta\alpha}\,{}_{(\ell)}\dot{\kappa}_{\alpha\beta} + {}_{(\ell)}\tilde{q}^{\alpha}\,{}_{(\ell)}\dot{\gamma}_{\alpha}]\,\mathrm{d}\mathscr{A}. \tag{199}$$

5.5. *Constitutive relations*

For layer (ℓ), we employ the constitutive relations

$$ {}_{(\ell)}\tilde{n}^{\alpha\beta} = \frac{{}_{(\ell)}E_{(\ell)}H}{1-({}_{(\ell)}\nu)^2}\,{}_{(\ell)}H^{\beta\alpha\gamma\delta}\,{}_{(\ell)}\varepsilon_{\gamma\delta}, \tag{200}$$

$$ {}_{(\ell)}\tilde{m}^{\alpha\beta} = \frac{{}_{(\ell)}E({}_{(\ell)}H)^3}{12[1-({}_{(\ell)}\nu)^2]}\,{}_{(\ell)}H^{\beta\alpha\gamma\delta}\,{}_{(\ell)}\rho_{\gamma\delta}, \tag{201}$$

$$ {}_{(\ell)}\tilde{q}^{\alpha} = K_{(\ell)}G_{(\ell)}H_{(\ell)}A^{\alpha\beta}\,{}_{(\ell)}\delta_{\beta}, \tag{202}$$

where

$$ {}_{(\ell)}H := {}_{(\ell)}h^{+} + {}_{(\ell)}h^{-}, \quad {}_{(\ell)}A^{\alpha\beta} := {}_{(\ell)}\mathbf{A}^{\alpha}\cdot{}_{(\ell)}\mathbf{A}^{\beta}, \tag{203}$$

is the total thickness of layer (ℓ), and the dual metric tensor of layer (ℓ) in the reference configuration. Whereas ${}_{(\ell)}E$, ${}_{(\ell)}G$, ${}_{(\ell)}\nu$ are the Young's modulus, shear modulus and Poisson's ratio for layer (ℓ), respectively, and K is the shear correction coefficient, when $K = 5/6$ eqn (202) is the same as that given in Naghdi (1972, p. 587). The elastic constant ${}_{(\ell)}H^{\beta\alpha\gamma\delta}$ given as follows

$$ {}_{(\ell)}H^{\beta\alpha\gamma\delta} = {}_{(\ell)}\nu_{(\ell)}A^{\beta\alpha}\,{}_{(\ell)}A^{\gamma\delta} + \tfrac{1}{2}(1-{}_{(\ell)}\nu)({}_{(\ell)}A^{\beta\gamma}\,{}_{(\ell)}A^{\alpha\delta} + {}_{(\ell)}A^{\beta\delta}\,{}_{(\ell)}A^{\alpha\gamma}), \tag{204}$$

is a fourth order elasticity tensor.

6. CLOSURE

In this paper, we have presented a detailed development of the dynamics of the geometrically-exact sandwich shell theory obtained by reduction of the general three-dimensional continuum mechanics theory by means of the one inextensible director assumption, and by using the balance of power of the sandwich shell. The fully nonlinear equations of motion is subsequently linearized systematically, based on the small deformation assumption. In particular, for the symmetric sandwich plate, the linearized equations can be reduced exactly to the classical results as given in Yu (1959). The equations of motion can also reduce exactly to the equations of motion for sandwich beams presented in Vu-Quoc and Ebcioğlu (1995), for which the numerical justification is given in Vu-Quoc and Deng (1995c). Our future work will be concentrated on the weak form and the finite element implementation of the equations of motion herein.

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